THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-eighth meeting of the Society was held in New York City on Saturday, February 28, 1914. The attendance at the two sessions included the following forty-two members:

Mr. H. Bateman, Mr. R. D. Beetle, Professor W. J. Berry, Professor Pierre Boutroux, Professor A. B. Coble, Dr. Emily Coddington, Professor F. N. Cole, Dr. J. R. Conner, Dr. G. M. Conwell, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. S. Franklin, Mr. H. Galajikian, Dr. G. H. Graves, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor E. R. Hedrick, Professor E. V. Huntington, Mr. S. A. Joffe, Professor C. J. Keyser, Mr. B. E. Mitchell, Mr. G. W. Mullins, Mr. J. A. Northcott, Dr. H. W. Reddick, Professor L. W. Reid, Professor L. P. Siceloff, Professor D. E. Smith, Professor Virgil Snyder, Mr. J. M. Stetson, Professor W. B. Stone, Professor H. S. White, Miss E. C. Williams, Professor A. H. Wilson, Professor E. B. Wilson, Professor F. S. Woods, Professor J. W. Young.

Vice-President L. P. Eisenhart occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. E. W. Castle, Princeton University; Professor P. J. Daniell, Rice Institute; Mr. L. R. Ford, Harvard University; Mr. C. M. Hill, State Normal School, Springfield, Mo.; Dr. R. A. Johnson, Adelbert College; Dr. L. M. Kells, Columbia University; Dr. W. W. Küstermann, Pennsylvania State College; Professor J. F. Reilly, State University of Iowa; Professor F. B. Williams, Clark University. Nine applications for membership in the Society were received.

An amendment of the Constitution of the Society was adopted by which the Secretary of the Chicago Section becomes ex officio a member of the Council.

The following papers were read at this meeting:

(1) Mr. H. S. VANDIVER: "Note on Fermat's last theorem."

(2) Dr. G. M. GREEN: "One-parameter families of space curves, and conjugate nets on a curved surface."

(3) Dr. G. M. CONWELL: "Brachistochrones under the action of gravity and friction."

(4) Mr. R. D. BEETLE: "A formula in the theory of surfaces."

(5) Dr. C. A. FISCHER: "The Legendre condition for a minimum of a double integral, with an isoperimetric condition."

(6) Mr. A. R. SCHWEITZER: "A generalization of functional equations."

(7) Mr. A. R. SCHWEITZER: "Some critical remarks on analytical realism."

(8) Professor E. V. HUNTINGTON: "A graphical solution of a problem in geology."

(9) Mr. H. GALAJIKIAN: "A relation between a certain non-linear Fredholm equation and the linear equation of the first kind."

(10) Dr. DUNHAM JACKSON: "Note on rational functions of several complex variables."

(11) Professor E. B. WILSON: "Infinite regions in geometry."

(12) Mr. H. BATEMAN: "The structure of the æther."

(13) Professors F. R. SHARPE and VIRGIL SNYDER: "Birational transformations of certain quartic surfaces."

(14) Professor F. R. SHARPE and Dr. C. F. CRAIG: "An application of Severi's theory of a basis to the Kummer and Weddle surfaces."

(15) Mr. B. E. MITCHELL: "Complex conics and their real representation."

(16) Professor W. H. ROEVER: "Analytic derivations of formulas for the deviations of falling bodies."

In the absence of the authors, the papers of Mr. Vandiver, Mr. Schweitzer, Dr. Jackson, Professor Sharpe and Dr. Craig, and Professor Roever were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. If $x^p + y^p + z^p = 0$ is satisfied in integers not divisible by the odd prime p, and $q(a) = (a^{p-1} - 1)/p$, Mr. Vandiver shows, by means of theorems due to Furtwängler and Mirimanoff, that one of the two following sets of conditions holds:

$$q(2) \equiv 0 \pmod{p^2} \qquad q(3) \equiv 0 \pmod{p}$$

or

$$q(2) \equiv q(3) \equiv q(5) \equiv 0 \pmod{p};$$

and if $p \equiv 2 \pmod 3$,

$$q(7) \equiv 0 \pmod{p}$$

also.

2. In this paper Dr. Green continues the study of a configuration which has already been the subject of a paper under a similar title, an abstract of which appeared in the January BULLETIN, page 171. In the present paper, a canonical development in the neighborhood of a point is found for a surface referred to a conjugate net; this development is

$$z = x^2 + y^2 + I^{(30)}x^3 + I^{(03)}y^3 + I^{(40)}x^4 + I^{(31)}x^3y + I^{(13)}xy^3 + I^{(04)}y^4 + \cdots,$$

where all the coefficients $I^{(jk)}$ are absolute projective invariants of the conjugate net. A canonical development

$$y = t + \theta^{(30)}x^3 + \theta^{(40)}x^4 + \cdots + \{\theta^{(12)}t^2 + \theta^{(13)}t^3 + \cdots\}x + \{\theta^{(21)}t + \theta^{(22)}t^2 + \cdots\}x^2 + \{\theta^{(31)}t + \cdots\}x^3 + \cdots,$$

$$z = t^2 + \varphi^{(03)}t^3 + \varphi^{(04)}t^4 + \cdots + \varphi^{(20)}x^3 + \varphi^{(40)}x^4 + \cdots + \{\varphi^{(13)}t^3 + \cdots\}x + \{\varphi^{(22)}t^2 + \cdots\}x^2 + \{\varphi^{(13)}t^3 + \cdots\}x^3 + \cdots$$

is also found for the one-parameter family of curves $t = \text{const.}$, the coefficients θ and φ being absolute projective invariants of the family.

3. Dr. Conwell obtains the parametric equations for the curve of quickest descent in a vertical plane when the coefficient of friction between the moving particle and the curve is a given constant.

4. Mr. Beetle proves the formula

$$\frac{1}{R^2} + \frac{1}{T^2} = \frac{K_m}{R} - K,$$

where K and K_m are the total and mean curvature of a surface, and $1/R$ and $1/T$ are the normal curvature and geodesic torsion of a curve on the surface. The geometric properties of the circle, the locus of the point whose cartesian coordinates are $(1/R, 1/T)$, furnish very elementary proofs of a number of well-known theorems, and also suggest several new theorems.

5. In this paper Dr. Fischer has derived the necessary condition for a maximum or a minimum of a double integral, relative to those surfaces over which a second double integral has a fixed value, which is analogous to the Legendre condition for a minimum of a simple definite integral.

6. Mr. Schweitzer proves the following theorem: If $f\{y, f(x, z)\} = f\{z, f(x, y)\}$, $f\{\phi(x, y), \psi(x)\} = \theta(y)$, $\phi\{f(x, y), \psi^{-1}(y)\} = \theta(x)$, $f\{\psi(x), \psi(y)\} = f(x, y)$, and $\phi\{\theta(x), y\} = \theta\{\phi(x, y)\}$ then there exists a function $\alpha(x)$ such that $\alpha\{\theta(x)\} = \alpha(x) + c_1$, $\alpha\{\psi(x)\} = \alpha(x) + c_2$, $\alpha\{f(x, y)\} = \alpha(x) - \alpha(y)$, $\alpha\{\varphi(x, y)\} = \alpha(x) + \alpha(y) + c_1 + c_2$. For $c_1 = 0$, that is, $\theta(x) = x$, this theorem reduces essentially to a theorem previously obtained by the author.

7. Mr. Schweitzer's second paper has appeared in the March number of *The Journal of Philosophy, Psychology and Scientific Methods*.

8. A common problem in geology is the determination of the relative position of two portions of a vein which have been separated by a fault. If A = the strike angle (that is, the angle between the horizontal traces of the fault plane and the vein plane); D = the dip of the fault (that is, the angle which the fault plane makes with the horizontal); V = the dip of the vein; and θ = the fault angle (that is, the angle in the fault plane from the strike of the fault to the line of intersection of the fault plane and the vein plane); then these four quantities are connected by the equation

$$\operatorname{ctn} \theta = \operatorname{ctn} A \cos D + \csc A \sin D \operatorname{ctn} V.$$

Professor Huntington's paper gives a very simple graphical representation of this equation by which θ can be readily determined without computation by drawing a straight line on a diagram. The theory of the method is simpler than

that given in Runge's Graphical Methods, since it does not require the use of line coordinates. The paper gives also the most general type of equation in four variables to which the straight line method of representation can be applied.

9. In this paper Mr. Galajikian shows that the non-linear Fredholm equation

$$u(s) = f(s) + \int_a^b \phi(s, t, u(t))dt,$$

where ϕ is a polynomial of the nth degree in u whose coefficients—continuous in s and t; $a < s < b$, $a < t < b$—form a set of symmetric and orthogonal functions, can be reduced, by a simple process, to a linear Fredholm equation of the first kind. The principle of reciprocity and hence the iterated kernel play an important part in the process.

10. Dr. Jackson proves the following theorem: Let the n arguments of the function $f(x_1, x_2, \cdots, x_n)$ be divided into k sets $x_1, \cdots, x_{r_1}; x_{r_1+1}, \cdots, x_{r_2}; \cdots; x_{r_{k-1}+1}, \cdots, x_n$. Let f be expressible in the neighborhood of every finite point of the space of the n variables as the quotient of two functions, each analytic at the point; and let f be expressible as the quotient of two power series in $x_1/x_{r_1}, x_2/x_{r_1}, \cdots, x_{r_1-1}/x_{r_1}, 1/x_{r_1}; x_{r_1+1}/x_{r_2}, \cdots, x_{r_2-1}/x_{r_2}, 1/x_{r_2}; \cdots; x_{r_{k-1}+1}/x_n, \cdots, x_{n-1}/x_n, 1/x_n$, for all sufficiently small values of these arguments. Then f is a rational function of $x_1, \cdots, x_n$.

The theorem for the case $k = n$, which was proved by Hurwitz, is assumed at the outset, and the demonstration then consists merely in showing that the method by which Professor Osgood (*Transactions*, 1912) recently made the passage from the case $k = n$ to the case $k = 1$, yields also the present more general result.

11. Professor Wilson desires to call attention to his view that ordinarily a geometry, as a geometrical or logical system in contradistinction to some algebraic representation of the system, has no infinite region; in particular that projective geometry and the projective plane have no infinite elements, and the same for circular geometry and its plane. The paper will appear in the BULLETIN.

12. Sir J. J. Thomson has developed a theory of the structure of the electric field in which an elementary electric charge has a single Faraday tube attached to it, through the agency of which energy is radiated and forces are exerted upon other electric charges. In Mr. Bateman's paper it is shown that the idea of an æther which consists of a network of threads or tubes is suggested by certain solutions of the fundamental equations

$$\text{rot } M = \mp \frac{i}{c}\frac{\partial M}{\partial t}, \quad \text{div } M = 0, \qquad M = H \pm iE.$$

A solution of the first type is obtained by putting

$$f(\alpha, \beta) \cdot M_x = \frac{\partial(\alpha, \beta)}{\partial(y, z)} = \pm \frac{i}{c}\frac{\partial(\alpha, \beta)}{\partial(x, t)},$$

. . . . . .

where α, β are defined by the equations

$$[x - \xi(\alpha, \beta)]^2 + [y - \eta(\alpha, \beta)]^2 + [z - \zeta(\alpha, \beta)]^2 = c^2[t - \tau(\alpha, \beta)]^2,$$

$$l(\alpha, \beta)[x - \xi(\alpha, \beta)] + m(\alpha, \beta)[y - \eta(\alpha, \beta)] + n(\alpha, \beta)[z - \zeta(\alpha, \beta)] = c^2 p(\alpha, \beta)[t - \tau(\alpha, \beta)],$$

$$l^2 + m^2 + n^2 \equiv c^2 p^2,$$

and involve the ambiguity $\pm$. When l, m, n, p are complex functions of the type $u(\alpha, \beta) + iv(\alpha, \beta)$ the existence of a network of threads is suggested by the properties of the functions α, β. The functions f, ξ, η, ζ, τ, l, m, n are arbitrary. Other types of electromagnetic fields associated with the network are obtained and a theory of the structure of the æther is developed in detail.

13. The object of the paper of Professors Sharpe and Snyder is to determine the necessary and sufficient conditions for the existence of birational transformations which leave quartic surfaces invariant that have all their genera equal to unity. Besides the case in which the surface contains a hyperelliptic sextic curve of genus 2, all the surfaces found have a pencil of elliptic or of rational curves, but the converse is not true. One exception is furnished by the surface containing a straight line, and otherwise unrestricted, and another is that of a quartic having a straight line that is an inflexional tangent of

every plane cubic whose plane passes through the line. The methods of Severi are used throughout, but a geometric interpretation is given for every symbolic equation.

14. In a series of papers F. Severi* has developed a theory by means of which any linear system of algebraic curves on an algebraic surface can be expressed linearly in terms of a finite number of such systems. In this paper Professor Sharpe and Dr. Craig apply this method to the study of certain birational transformations which leave the Kummer and Weddle surfaces invariant. In particular the question of the periodicity of the product of two such transformations is studied. Some new relations among the different types of transformations are obtained.

15. Following the method of Laguerre, amplified and extended by Study, Mr. Mitchell considers the real representation of the complex conic. By the term complex conic is meant the geometric configuration associated with the general equation of the second degree in two variables, the variables and the five constants being complex quantities. Since the method of approach is somewhat different from that of Study, the complex line and circle are employed for purposes of introduction and for methods of procedure in the case proper.

Considerable reduction and simplification is effected by identifying these configurations as members of complex pencils, meaning by complex pencil one with real bases and complex parameter. By selecting certain new bases (real) the equations of the complex line, complex circle, and complex conic reduce as follows:

(1) The equation of the complex line to a form with one pure imaginary coefficient, as Study found.

(2) The equation of the complex circle to a form with one complex coefficient, namely, a particular value of the parameter of the pencil of which it is a member.

(3) The equation of the complex conic to a form with two complex constants, namely, the coefficient of the product term and the constant term, the former being the parameter of the pencil of which the conic is a member, the latter a linear function of the parameter.

* See F. Severi, "Complementi alla teoria della base per la totalità delle curve di una superficie algebrica," *Circ. Mat. Palermo, Rendiconti*, vol. 30 (1911), pp. 265–288 and the papers there referred to.

A single transformation picturing the complex points on any one of these curves is of course not involutorial; if, however, the transformations belonging to a curve and its conjugate are applied consecutively, the points of the plane are left in place, the involutorial property thereby being included, since the real curve is its own conjugate.

In the representation of all three cases a parameter plane (not connected with the pencil) is employed for purposes of simplification.

Some interesting results arise from superposing the two picture planes upon the cartesian plane in which the curve lies.

16. The formulas (3) and (4) here derived by Professor Roever have already been derived by him by a quasi-geometric method. (See *Transactions*, volume 13, number 4.) To define the deviations, he denotes by P_1 a point fixed with respect to the earth, and by $P_1\eta$, $P_1\xi$, $P_1\zeta$ axes directed to the east, south, and zenith, respectively. A particle falling from the point P_0, which is at distance h above P_1 and in the vertical $P_1\zeta$ of P_1, meets the plane $P_1\eta\xi$ in a point C. The *easterly* and *southerly deviations*, denoted by E. D. and S. D., are the distances of C to the east of $P_1\xi$ and to the south of $P_1\eta$, respectively. The equations of motion are

$$\begin{aligned} &\xi'' - 2\omega \sin \phi_1 \cdot \eta' = \partial W/\partial \xi, \\ (1)\qquad &\eta'' + 2\omega(\sin \phi_1 \cdot \xi' + \cos \phi_1 \cdot \zeta') = \partial W/\partial \eta, \\ &\zeta'' - 2\omega \cos \phi_1 \cdot \eta' = \partial W/\partial \zeta, \end{aligned}$$

where accents denote derivatives with respect to the time t, ω the angular velocity of the earth, ϕ_1 the astronomical latitude of P_1, and W the potential function of weight field of force. (It is important to note that these equations implicitly contain ω^2, which is involved in the expression of W). The solution of (1) which is subject to the initial conditions: when $t = 0$, $\eta = \xi = 0$, $\zeta = h$, $\eta' = \xi' = \zeta' = 0$, is of the form

$$\begin{aligned} &\eta = a_2t^2 + a_3t^3 + \cdots, \\ (2)\qquad &\xi = b_2t^2 + b_3t^3 + b_4t^4 + \cdots, \\ &\zeta = h + c_2t^2 + \cdots, \end{aligned}$$

where the coefficients a_2, b_2, c_2, a_3, etc., involve h. If $\bar{t}$ denotes the time of fall, then for $t = \bar{t}$, $\zeta = 0$, $\eta =$ E. D., $\xi =$ S. D., and from (2) there results

$$
\begin{aligned}
h &= -\tfrac{1}{2}W_\zeta^{(1)} \cdot \bar{t}^2 + \tfrac{1}{24}W_\zeta^{(1)}[5W_{\zeta\zeta}^{(1)} + 4\omega^2\cos^2\phi_1]\bar{t}^4 \\
&\qquad + \tfrac{1}{6}\omega W_\zeta^{(1)}W_{\eta\zeta}^{(1)}\cos\phi_1 \cdot \bar{t}^5 + \cdots, \\
\text{E. D.} &= -\tfrac{1}{3}\omega\cos\phi_1 \cdot W_\zeta^{(1)} \cdot \bar{t}^3 - \tfrac{5}{24}W_\zeta^{(1)} \cdot W_{\eta\zeta}^{(1)} \cdot \bar{t}^4 \\
&\qquad + \tfrac{1}{60}\omega W_\zeta^{(1)}[9\sin\phi_1 \cdot W_{\xi\zeta}^{(1)} + \cos\phi_1 \cdot (9W_{\zeta\zeta}^{(1)} \\
&\qquad\qquad - W_{\eta\eta}^{(1)} + 4\omega^2)]\bar{t}^5 + \cdots, \\
\text{S. D.} &= -\tfrac{1}{24}W_\zeta^{(1)}[5W_{\xi\zeta}^{(1)} + 4\omega^2\sin\phi_1\cos\phi_1]\bar{t}^4 \\
&\qquad - \tfrac{1}{60}\omega W_\zeta^{(1)}[9\sin\phi_1 \cdot W_{\eta\zeta}^{(1)} + \cos\phi_1 \cdot W_{\xi\eta}^{(1)}]\bar{t}^5 + \cdots.
\end{aligned} \tag{3}
$$

The elimination of $\bar{t}$ from these relations yields the formulas

$$
\begin{aligned}
\text{E. D.} &= \tfrac{2}{3}\sqrt{2}\,\omega\cos\phi_1 \cdot \frac{h^{\frac{3}{2}}}{g_1^{\frac{3}{2}}} - \frac{5}{6}\left(\frac{\partial g}{\partial \eta}\right)_1\frac{h^2}{g_1} \\
&\qquad + \frac{\sqrt{2}}{30}\omega\left\{18\sin\phi_1 \cdot \left(\frac{\partial g}{\partial \zeta}\right)_1 - \cos\phi_1 \cdot \left[7\left(\frac{\partial g}{\partial \zeta}\right)_1\right.\right. \\
&\qquad\qquad \left.\left. - 2\left(\frac{\partial W_\eta}{\partial \eta}\right)_1 + 4\omega^2(5\cos^2\phi_1 - 2)\right]\right\}\frac{h^{\frac{5}{2}}}{g_1^{\frac{3}{2}}}, \\
\text{S. D.} &= \frac{1}{6}\left[4\omega^2\sin\phi_1\cos\phi_1 - 5\left(\frac{\partial g}{\partial \xi}\right)_1\right]\frac{h^2}{g_1} \\
&\qquad + \frac{\sqrt{2}}{15}\omega\left[\cos\phi_1 \cdot \left(\frac{\partial W_\xi}{\partial \eta}\right)_1 - 9\sin\phi_1 \cdot \left(\frac{\partial g}{\partial \eta}\right)_1\right]\frac{h^{\frac{5}{2}}}{g_1^{\frac{3}{2}}}.
\end{aligned} \tag{4}
$$

In formulas (3) and (4), $W_\zeta^{(1)}$, $W_{\xi\zeta}^{(1)}$, etc., stand for the values of $\partial W/\partial\zeta$, $\partial^2 W/\partial\xi\partial\zeta$, etc., at the point P_1, and $g_1 = -W_\zeta^{(1)}$ is the value, at P_1, of the acceleration due to weight. If in formulas (3) and (4) the coefficients be expressed in terms of the astronomical latitude ϕ_0 of P_0 and the values, at P_0, of the derivatives of W with respect to the variables η, ξ, ζ (which are measured to the east, south, and zenith, at P_0), formulas I and III of the author's *Transactions* paper are obtained, where however $\bar{\eta}$, $\bar{\xi}$, $\bar{\zeta}$, ϕ, t, η, ξ, ζ stand for E. D., S. D., $-h$, ϕ_0, $\bar{t}$, $\dot{\eta}$, $\dot{\xi}$, $\dot{\zeta}$, which are here used. See also BULLETIN, volume 20, number 4, page 175.

F. N. COLE,
Secretary.

A NON-ENUMERABLE WELL-ORDERED SET.

BY PROFESSOR A. B. FRIZELL.

(Read before the American Mathematical Society, December 27, 1913.)

THE article "Intuitionism and formalism" in the November BULLETIN (pages 81–96) raises the question whether the field of denumerably infinite processes is a closed domain. The present note, based on a paper which the writer presented to the Cambridge Congress under the title "Axioms of ordinal magnitudes,"* undertakes to show that this question must be answered in the negative.

It is admitted by both formalist and intuitionist that the natural numbers rest on valid assumptions. In other words, it is legitimate to postulate two rules of combination—let us call them "lower" and "higher"—whereof the higher is to be distributive over the lower, a modulus for the higher rule, and the totality of symbols obtainable therefrom by the "axioms of numbering"† in the paper just mentioned.

Let us drop the postulate of the modulus, leaving λ with no properties except that it is to precede every other element of the set. The higher rule now generates automatically the ω-series

$$\lambda_2' = \lambda \square \lambda, \quad \lambda_2'' = \lambda \square \lambda_2', \quad \cdots,$$

while the lower rule yields not only a similar set

$$\lambda_1' = \lambda \circ \lambda, \quad \lambda_1'' = \lambda \circ \lambda_1', \quad \cdots,$$

but also other such sets built up on other members of the first

$$\lambda_{(2,1}^{(n,2} = \lambda_2^{(n} \circ \lambda_2^{(n}, \quad \lambda_{(2,1}^{(n,3} = \lambda_2^{(n} \circ \lambda_2^{(n,2}, \quad \cdots \quad (n = 1, 2, \cdots)$$

and sets of polynomial elements.

This process makes no use of the distributive law; the same result would be reached by postulating two rules totally unconnected except by the relation of order. And if it is per-

* Proceedings of the Fifth International Congress of Mathematicians, pp. 327–336.

† Loc. cit., p. 328. A numbered set is an ω-series of symbols generated by the law that if κ belongs to the set so does $\lambda \circ \kappa$ (resp. $\kappa \circ \lambda$) also, where λ is an "ausgezeichnetes Element." The set is ordered so that $\kappa < \lambda \circ \kappa$, resp. $\kappa < \kappa \circ \lambda$.

missible to postulate two rules entirely independent of one another, then we are also entitled to an ω-series of unrelated rules together with all symbols that can be generated from λ by applying these rules in accordance with the axioms of numbering. Such a set is postulated in the "transfinite axioms"* of the paper already quoted.

It is proposed to prove that this set of symbols cannot be put into one-to-one correspondence with the natural numbers. It is sufficient to consider polynomials like $\omega_{(10}^{19} + \omega_{(9}^{250} + w_{(8}^{101} + \cdots + \omega_{(2}^{83}$, i. e., those formed by the lowest rule and having every term a primitive element. Suppose this set denumerable so that we may write them: $P_1, P_2, \cdots, P_n, \cdots$. Denote by p_1 the lowest term in P_1, by p_2 the next to lowest term in $P_2, \cdots$, by p_n the nth lowest in P_n. Let the corresponding upper indices be $i_1, i_2, \cdots, i_n$ so that $p_n = \omega_{(\nu}^{i_n}$ and write $i_n = 0$ if P_n has less than n terms. Form a new polynomial by changing these indices to $\kappa_1 \neq i_1, \kappa_2 \neq i_2, \cdots, \kappa_n \neq i_n$. Either it is possible in this way to get a polynomial not in the ω-series $P_1, P_2, \cdots$ or it is not. The first alternative contradicts the assumption of denumerability. The second implies polynomials with an actually infinite number of terms, admitting each an ω-series of upper indices. Such a "Belegung," however, is a continuum† and hence not countable. Thus in either case the supposition in false, and the theorem is proved.

Thus by proceeding strictly in accord with the demands of neo-intuitionism, using only the "bare two-oneness" to which this philosophy stands committed, we are inevitably led to a concept which it refuses to recognize. This, of course, need not disturb an intuitionist who cuts loose from the principium contradictionis, but it is matter for regret to find formalism attacked by aid of the principium on the grounds of the Burali-Forti "paradox," which is nothing more than the logical freak of the illusory self-contradictory concept.‡

McPherson, Kans.

* Loc. cit., p. 329. These symbols fall into three classes: (1) *primitive* elements, of the type $\omega_{(n}^{i}$; (2) *monomial* elements, given by the formula $\omega_{(\nu_1, \nu_2, \cdots \nu_n)}^{(i_1, i_2, \cdots i_n)}$ $(\nu_1 > \nu_2 > \cdots)$; (3) *polynomials*, or combinations of monomials by lower rules than those used in any individual term.

† There really is a continuum contained in the set of symbols postulated by axioms T^{ω_1} of the paper previously cited (p. 333), since every element of this set involves an actual infinity of "Bestimmungstücke."

‡ For example, "the totality of self-contradictory concepts" is a self-contradictory concept, because it does not include the concept "a self-contradictory concept not contained in the totality of self-contradictory concepts"; but the latter is a self-contradictory concept.

NOTE ON THE FREDHOLM DETERMINANT.

BY DR. W. A. HURWITZ.

(Read before the American Mathematical Society, December 30, 1913.)

THE theory of linear integral equations presents many analogies with the theory of linear algebraic equations; in fact the former may be regarded in a quite definite and accurate sense as a limiting case of the latter. As in various other mathematical theories concerned with limiting cases, two methods suggest themselves for the proof of theorems: one may go through the process of taking limits in the results of the algebraic theory—this method is typified by the early work of Hilbert* on integral equations; or one may use the algebraic theory merely to suggest theorems, which one then proves independently—this is the procedure in the fundamental paper of Fredholm.† While any sweeping statement comparing the two plans would be unwise, it seems reasonably clear that the second method will ordinarily be the more elegant.

In the course of a detailed study of certain properties of the kernel of an integral equation, Platrier‡ has recently given an expression for all higher minors of the determinant for any kernel in terms of the first minor and the determinant itself; his proof consists in carrying over by an accurate limit process the corresponding theorem for algebraic determinants. The same theorem had suggested itself to the present writer in studying mixed linear integral equations; as the proof devised by him, following directly from well known facts about the minors and involving no new passage to the limit, seems rather simpler, its presentation is worth while, in view of the importance of the theorem.§

It is assumed that the kernel $K(s, t)$ is a continuous function, $a \leqq s \leqq b$, $a \leqq t \leqq b$; integration always takes place between the limits a, b, which will not be written. The

* *Göttinger Nachrichten*, 1904, p. 4.

† *Acta Mathematica*, vol. 27 (1903), p. 365.

‡ *Liouville's Journal*, vol. 9 (1913), p. 249, equations (30), (31).

§ The writer learns that direct proofs of the theorem have also been found by Dr. T. H. Hildebrandt.

minors satisfy the identity (corresponding to the expansion of minors of an algebraic determinant according to columns)*

$$\begin{aligned} &D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m) \\ &\quad = K(s_1, t_1)D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m) \\ &\qquad - K(s_1, t_2)D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m) \\ &\qquad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\qquad + \lambda \int K(s_1, r)D(\lambda; r, s_2, \cdots, s_m; t_1, t_2, \cdots, t_m)dr. \end{aligned}$$

Regarding all the quantities s, t, λ as constant except s_1, we have here an integral equation of the form

$$u(s_1) = f(s_1) + \lambda \int K(s_1, r)u(r)dr;$$

whether or not $D(\lambda) = 0$, we deduce that

$$D(\lambda)u(s_1) = D(\lambda)f(s_1) + \lambda \int D(\lambda; s_1, r)f(r)dr.$$

Thus

$$\begin{aligned} &D(\lambda)D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m) \\ &\quad = [D(\lambda)K(s_1, t_1) + \lambda \int D(\lambda; s_1, r)K(r, t)dr] \\ &\qquad\qquad \times D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m) \\ &\qquad - [D(\lambda)K(s_1, t_2) + \lambda \int D(\lambda; s_1, r)K(r, t_2)dr] \\ &\qquad\qquad \times D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m) \\ &\qquad + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Simplifying each expression in brackets by the identity

$$D(\lambda)K(s, t) + \lambda \int D(\lambda; s, r)K(r, t)dr = D(\lambda; s, t),$$

we have

$$\begin{aligned} &D(\lambda)D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m) \\ &\quad = D(\lambda; s_1, t_1)D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m) \\ &\qquad - D(\lambda; s_1, t_2)D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m) \\ &\qquad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{1}$$

The theorem to be proved now follows at once. In (1)

* Horn, Einführung in die Theorie der linearen partiellen Differentialgleichungen, p. 202.

take $m = 2$; we find

$$D(\lambda)D(\lambda; s_1, s_2; t_1, t_2)$$
$$= D(\lambda; s_1, t_1)D(\lambda; s_2, t_2) - D(\lambda; s_1, t_2)D(\lambda; s_2, t_1)$$
$$= \begin{vmatrix} D(\lambda; s_1, t_1) & D(\lambda; s_2, t_1) \\ D(\lambda; s_1, t_2) & D(\lambda; s_2, t_2) \end{vmatrix}.$$

This gives an expression for the minor of second order. Write (1) for $m = 3$, multiply by $D(\lambda)$, and replace the minor of second order by the value just obtained; we have a form for the minor of third order. Repeating this process, we find by mathematical induction the general formula

$$[D(\lambda)]^{m-1}D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)$$
$$= \begin{vmatrix} D(\lambda; s_1, t_1) & D(\lambda; s_2, t_1) & \cdots & D(\lambda; s_m, t_1) \\ D(\lambda; s_1, t_2) & D(\lambda; s_2, t_2) & \cdots & D(\lambda; s_m, t_2) \\ \cdot & \cdot & \cdot & \cdot \\ D(\lambda; s_1, t_m) & D(\lambda; s_2, t_m) & \cdots & D(\lambda; s_m, t_m) \end{vmatrix}, \tag{2}$$

which is the relation established by Platrier. In case $D(\lambda) \neq 0$ an equivalent form is of course obtained, as he points out, on dividing both sides by $[D(\lambda)]^{m-1}$,

$$D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)$$
$$= D(\lambda) \begin{vmatrix} k(\lambda; s_1, t_1) & k(\lambda; s_2, t_1) & \cdots & k(\lambda; s_m, t_1) \\ k(\lambda; s_1, t_2) & k(\lambda; s_2, t_2) & \cdots & k(\lambda; s_m, t_2) \\ \cdot & \cdot & \cdot & \cdot \\ k(\lambda; s_1, t_m) & k(\lambda; s_2, t_m) & \cdots & k(\lambda; s_m, t_m) \end{vmatrix}, \tag{3}$$

where $\lambda k(\lambda; s, t)$ is the resolvent to the kernel $\lambda K(s, t)$.

It is perhaps not superfluous to remark that the use of (2) as the *definition* of higher minors would lead to little or no simplification of the Fredholm theory. It would be necessary to show that the higher minors as thus defined are integral functions of the complex variable λ; the proof (thus required) that the m-rowed determinant, whose elements are values of the first minor, must possess at any principal value a zero of order $\geqq m - 1$, while not very difficult, is sufficiently so to preclude any substantial advantage in this mode of approach.

CORNELL UNIVERSITY,
December, 1913.

TIME AS A FOURTH DIMENSION.

BY PROFESSOR R. C. ARCHIBALD.

WHO first conceived the idea of time as a fourth dimension? In view of the recent development of the theory of relativity the answer to this question is of some interest to both mathematician and physicist. With the definite formulation of our ideas concerning space of n dimensions during the latter part of the last century the conception occurred, doubtless, to many people. But, so far as I am aware, writers who discuss the subject have invariably given the credit to Lagrange (1736–1813), and have referred to a paragraph in either the first* (1797, Loria† and Enriques‡) or second§ (1813, Dühring‖ and Sommerville¶) editions of his Théorie des Fonctions analytiques.** This paragraph, the first of the section "Application de la théorie des fonctions à la mécanique," is as follows:††

"Nous allons employer la théorie des fonctions dans la mécanique. Ici les fonctions se rapportent essentiellement au temps, que nous désignerons toujours par t; et comme la position d'un point dans l'espace dépend de trois coordonnées rectangulaires x, y, z, ces coordonnées, dans les problèmes de mécaniques, seront censées être fonctions de t. Ainsi on peut regarder la mécanique comme une géométrie à quatre dimensions, et l'analyse mécanique comme une extension de l'analyse géométrique."

* Paris, Prairial, an V [= 1797], p. 223.

† G. Loria, Die hauptsächlichsten Theorien der Geometrie . . . ins Deutsche übertragen von F. Schütte, Leipzig, 1888, pp. 116–117; also Il Passato ed il Presente delle principali Teorie geometriche, terza ed., Torino, 1907, p. 327.

‡ F. Enriques, Encyclopédie des Sciences Math., tome III, vol. 1, p. 71 (1911); also Encyklopädie d. Math. Wiss., Bd. III, p. 63 (1907).

§ Paris, 1813, p. 311.

‖ E. Dühring, Kritische Geschichte der allgemeinen Principien der Mechanik. Leipzig, 1877, p. 562f.

¶ D. M. Y. Sommerville, Bibliography of Non-Euclidean Geometry, London, 1911, p. 17.

** There are four other editions of this work: A German translation of the first by J. P. Grüson, Berlin, 1798–99; a German edition, with notes, of the second by A. L. Crelle in Bd. I of Lagrange's Mathematische Werke, Berlin, 1823; a fourth French edition reprinted from the second in Lagrange's Œuvres, tome IX (1881); the third French edition (best), edited by J. A. Serret, Paris, 1847.

†† I quote from the second edition. The only difference of the paragraph from that of the first edition is in the first sentence.

No further elaboration of this idea occurs in the published writings of Lagrange.

Apparently independent of earlier suggestion, a writer in *Nature** developed the idea of "time-space." A decade later the popular English novelist Herbert George Wells wrote a serial, for W. E. Henley's *New Review*,† in which an invention for negotiating similar "space" is basic.

Then came the important mathematical developments by R. Mehmke,‡ A. Einstein,§ H. Minkowski,‖ M. Abraham,¶ A. Sommerfeld,** Wilson and Lewis,†† to mention only a few names.‡‡

But the main purpose of this note is to make a very small contribution to the history of geometry by way of answer to the question with which this paper commences. Long before the publication of the Fonctions analytiques, indeed as far back as 1754, d'Alembert (1717–1783) published the article "Dimension" in the famous Encyclopédie edited by Diderot and himself.§§ Here the idea of fourth dimension is dwelt

* March 26, 1885, vol. 31, p. 481; "Four-dimensional space," by "S."

† Jan.–June, 1895, vol. 12; "The time machine," republished in book form, London, 1895.

‡ "Ueber die darstellende Geometrie der Räume von vier und mehr Dimensionen, mit Anwendungen auf die graphische Mechanik, die graphische Lösung von Systemen numerischer Gleichungen und auf Chemie." Vortrag, *Math. naturw. Mitt.*, Stuttgart (2), vol. 6, 1904, p. 44 ff.

§ "Zur Elektrodynamik bewegter Körper," *Annalen der Physik*, 1905, Bd. 322, p. 391 ff.

‖ "Raum und Zeit," Vortrag, gehalten zu Köln, am 21 September, 1908, Leipzig, 1909. Numerous editions of this Vortrag have been published. Another edition may be found in Gesammelte Abhandlungen von Hermann Minkowski, Bd. II, Leipzig u. Berlin, 1911, p. 431 ff.

¶ "Sull'elettrodinamica di Minkowski (Vettori e tensori di quattro dimensioni)," *Palermo, Rend. Circ. Mat.*, vol. 30 (1910), p. 33 ff.

** "Zur Relativitätstheorie. I. Vierdimensionale Vektoralgebra," *Annalen der Physik*, 1910, Bd. 337, p. 749 ff.

†† E. B. Wilson and G. N. Lewis, "The space-time manifold of relativity. The non-euclidean geometry of mechanics and electromagnetics," *Proc. Amer. Acad. of Arts and Sciences*, Nov., 1912, vol. 48, pp. 389–507.

‡‡ For bibliographies of the theory of relativity the reader may consult M. Laue, Das Relativitätsprinzip, 2. Aufl., Braunschweig, 1913, and the article by J. Laub in the *Jahrbuch der Radioaktivität und Elektrotechnik*, 1910, Bd. 7, p. 405 f.

§§ Encyclopédie ou Dictionnaire raisonné des sciences, des Arts et des métiers, par une societé de gens de Lettres. Mis en ordre et publié par M. Diderot . . .; et quant à la partie mathématique par M. d'Alembert . . ., tome IV, Paris, M. DCC. LIV, pp. 1009–1010; 3e éd., Livourne, tome IV, 1772, pp. 922–923; Encyclopédie méthodique, Mathématiques, par MM. d'Alembert, l'abbé Bossut, de la Lande, le Marquis de Condorcet, etc., tome I, p. 531, Paris, 1784. There are several other editions which I have not seen. The same article may be found in all the editions above specified. It is signed by "O."=d'Alembert.

upon more at length than by Lagrange, and d'Alembert attributes the conception of time as a fourth dimension to "un homme d'esprit de ma connaisance."

It seems questionable that Lagrange, a youth of 18 years, was characterized in this way, even though he was in the following year appointed professor of mathematics in the Artillery School at Turin. But at least credit given Lagrange in this connection must, in the future, be rendered with small show of authority.

To give appropriate setting to what d'Alembert writes, I quote somewhat fully* from his article in question:

". . . On se sert particulièrement du mot *dimension* pour exprimer les puissances des racines ou valeurs des quantités inconnues des équations, que l'on appelle les *dimensions* de ces racines. . . ."

"Ainsi, dans une équation simple du premier degré; la quantité inconnu n'a qu'une *dimension*, comme $x = a + b$. Dans une équation du second degré. . . . "

"En général on dit, *en Algèbre*, qu'une quantité comme *abcd*, *abc*, *ab*, etc. est d'autant de *dimensions* qu'il y a de lettres ou de facteurs dont elle est composée. Ainsi *abcd* est de quatre *dimensions*, *abc* de trois, etc. On sent assez la raison de cette denomination prise de la géométrie. Si, par example, les produisans ou facteurs *a*, *b*, *c* du produit *abc* sont representés par des lignes, le produit *abc* sera représenté par un solide ou parallelepipede (*sic*), dont l'une des dimensions est *a*, l'autre *b*, l'autre *c*; de même le produit *ab* est de deux *dimensions*, parce qu'il peut représenter une surface ou figure rectangle de deux *dimensions*, *a*, *b*, etc. Au reste, il ne peut y avoir proprement que des quantités de trois *dimensions;* car passé le solide, on n'en peut concevoir d'autres. Qu'est-ce donc que les quantités comme a^4, a^5, qu'on employe dans l'application de l'algèbre à la géométrie? Ces quantités peuvent être considérés sous deux points de vûe. Ou la ligne *a* est représentée par un nombre arithmétique, et en ce cas a^4 est le quatrieme (*sic*) puissance de ce nombre; ou bien on doit supposer a^4 divisé par une certaine ligne à volonté, qui réduise le nombre des dimensions à 3. Par example, soit $x^5 + ax^4$

In this connection reference should be given to the article "Abmessung" by G. S. Klügel in his Mathematisches Wörterbuch, I. Theil, Leipzig, 1803, where d'Alembert's ideas are reproduced.

* In making the quotation "et" has been substituted for "&."

$+ b^5 = 0$, je dis que cette équation est la même chose que $(x^5 + ax^4 + b^5)/c^2 = 0$, ce qui réduit les dimensions à trois."

"Remarquez qu'on peut toujours faire cette division; car, dans la géométrie, tout se réduit toujours à des équations. On ne considere (*sic*) a^4 que pour le comparer à quelque autre quantité de même *dimension;* et il est visible qu'une équation continue d'avoir lieu, lorsqu'on divise tous ses termes par une quantité constante quelconque. Ou bien on peut regarder a et b dans l'équation comme des nombres, qui soient entr'eux comme les lignes représentées par a et b, et alors x sera un nombre, et on n'aura que faire de division. Cette manière de considérer les quantités de plus de *trois dimensions*, est aussi exacte que l'autre; car les lettres algébriques peuvent toujours être regardées comme représentant des nombres, rationels ou non. J'ai dit plus haut qu'il n'étoit pas possible de concevoir plus de trois *dimensions*. Un homme d'esprit de ma connaissance croit qu'on pourroit cependant regarder la durée comme une quatrieme (*sic*) *dimension*, et que le produit du tems (*sic*) par la solidité, serait en quelque maniere (*sic*) un produit de quatre *dimensions;* cette idée peut être contestée, mais elle a, ce me semble, quelque mérite, quand ce ne servoit que celui de la nouveauté."

BROWN UNIVERSITY,
PROVIDENCE, R.I.
December 26, 1913.

THE DISCOVERY OF INVERSION.

Über Bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion. By F. BÜTZBERGER. B. G. Teubner, Leipzig, 1913. viii + 60 pp.

As indicated by the title, this interesting monograph consists of three chapters in which Professor Bützberger presents the results of a critical and historical investigation on bicentric polygons, Steinerian series of circles and spheres, and the discovery of inversion.

The first chapter treats of polygons which are simultaneously inscribed to one and circumscribed to the other of two non-intersecting circles in a plane. The conditions for closure of such polygons have been investigated by Poncelet, and it is for this reason that they are frequently called Poncelet's

polygons. Jacobi, as is well known, gave an elegant solution of Poncelet's problem by means of elliptic functions and this is still the best method not only for this but also for a great number of other problems of closure. Bützberger's method is very direct and elementary and aims to follow the trail which presumably led Steiner to the equations between the radii and the distance e between the centers of two circles with the property of closure, of which one with radius r encloses the other with radius ρ. In case of such enclosure the corresponding rectilinear polygons are called bicentric. When $e = 1$ and $r = 2$, the equations have the remarkable property that the sum of their coefficients is always unity. The equation for a polygon of eight sides, for example, is

$$8704\rho^8 - 23040\rho^6 + 36936\rho^4 - 29160\rho^2 + 6561 = 0,$$

and the sum of the coefficients is

$$52201 - 52200 = 1.$$

In the second chapter we find a discussion of Steiner's series of circles and spheres. Most of the problems connected with these series have been studied by a number of geometers and by various methods, and there is practically no new ground covered in this part of the book.

It seems that the author is not familiar with some recently established properties of closure in connection with linkages. For example, all problems concerning Poncelet's polygons (making use of inversion) and Steinerian series result as special cases from the theorem:

*If each pair of consecutive circles of a series of circles which all touch two circles C_1 and C_2, of which C_2 encloses C_1, intersect in two points B_i and B_i' and if the points B_1, B_2, B_3, $\cdots$ are on a circle C_3, then the points B_1', B_2', B_3', $\cdots$ are also on a circle C_4. If the series of circles closes and contains n circles, then every other series based upon the same circles C_1, C_2, C_3 closes and contains n circles.**

* This theorem was found by the writer in 1901. See "An application of elliptic functions to certain linkages," in *Annals of Mathematics*, 2d series, vol. 2; also "Applications of elliptic functions to problems of closure," *University of Colorado Studies*, vol. 1, pp. 81–133 (1902); and "Kinematische Gelenksysteme und die durch sie erzeugten geometrischen Transformationen," *Jahresbericht der Kantonsschule*, Solothurn (1907).

In an article "On circular transformations," *Annals of Mathematics*, vol. 12, pp. 141–160 (1899), most of the ordinary problems in Steinerian series are proved in a simple manner.

The third chapter, in which an account of the discovery of inversion (transformation by reciprocal radii) is given, is in our judgment the most important.

After the death of Steiner in 1863, his unpublished manuscripts, letters, and scientific fragments were deposited in the city library of Bern, where 30 years later Professor Graf rescued them from oblivion. He turned them over to Professor Bützberger for critical study and eventual publication.

As early as 1826 Steiner had nearly finished a manuscript of 360 pages, ready for print, on "Allgemeine Theorie des Berührens und Schneidens der Kreise in der Ebene, der Kugeln im Raum, und der Kreise auf der Kugelfläche, mit vielen neuen Sätzen und Untersuchungen in einem systematischen Entwicklungsgange dargestellt."

According to Bützberger, after the publication of this work and Steiner's explicit statement of the principle of inversion, dated February 8, 1824, and published on pages 50–55 of the work under review, there cannot be the slightest doubt that it was Steiner who first established and applied what Liouville in 1847 called the transformation by reciprocal radii. From the nature of a great number of remarkable theorems published by Steiner in the early volumes of *Crelle's Journal* it had long been suspected that Steiner was familiar with this transformation and made extended use of it. Plücker's insinuation in an article published in the same journal, where he gave an analytic proof of Steiner's famous solution of Malfatti's problem and its generalization, that Steiner had probably no proof for his construction, was manifestly unfair. It simply shows that Plücker had no conception and consequently no appreciation of the great power and originality of Steiner's methods. Steiner on the other hand disdained as "caricatures" some trivial applications which Plücker and also Magnus made of the principle of inversion.

In the following decades the same transformation was found independently by several investigators: By Bellavitis* in 1836, by J. W. Stubbs† and J. R. Ingram‡ in 1842 and 1843 and by William Thomson§ (Lord Kelvin) in 1845.

* *Annali delle Scienze*, Padova, vol. 17, p. 126.

† *Philosophical Magazine*, vol. 23, pp. 338–347. The title of the paper referred to is "On the application of a new method to the geometry of curves and curve surfaces." January 31, 1843.

‡ *Transactions of the Dublin Philosophical Society*, vol. 1, 1842–43.

§ *Journal de Mathématiques pures et appliquées*, vol. 10, pp. 364–367, "Extrait d'une lettre de M. William Thomson"; vol. 12, pp. 256–290, "Extrait de deux lettres addressées à M. Liouville," 1846.

The little book is carefully and clearly written and contains only a few minor errors. On page 58, second line from bottom, read "Nouvelle méthode" instead of "Nouvelles Méthodes." On the next page read J. W. instead of J. J. Stubbs. The reference to J. W. Stubbs and J. R. Ingram is ambiguous. In the *Philosophical Magazine* Stubbs does not make the least reference to Ingram, and evidently claims priority of the discovery.

ARNOLD EMCH.

SHORTER NOTICES.

Poliedri, Curve e Superficie secondo i Metodi della Geometria descrittiva. By GINO LORIA, professor of mathematics at the University of Genoa. Milan, Ulrico Hoepli, 1912. (Manuali Hoepli, nos. 148–149.) xv + 235 pages and 62 figures.

Vorlesungen über darstellende Geometrie. Zweiter Teil: Anwendungen auf ebenflächige Gebilde, Kurven und Flächen. By GINO LORIA. Autorisierte, nach dem italienischen Manuskript bearbeitete, deutsche Ausgabe, by FR. SCHÜTTE, teacher in the gymnasium at Düren. (Teubner's Sammlung, no. XXV_2.) Leipzig and Berlin, B. G. Teubner, 1913. xii + 294 pages and 146 figures.

THE present volume of Professor Loria's treatise presupposes a knowledge of the previous one (Metodi di Geometria descrittiva). It is concerned with the applications of the principles there developed to the graphical representation of polyhedra, curves, and surfaces. The style is clear and concise, and each step of a procedure is fully explained, but the theorems of analytical geometry and of the calculus that are made use of are stated without proof or reference, thus providing a series of statements having very questionable value. If the reader is already familiar with these theorems, they need not be repeated here; if he is not familiar with them, the words will convey little, if any, meaning to him.

The first chapter is concerned with the graphical solution of spherical triangles, incidentally including the derivation of the fundamental formulas of spherical trigonometry. The next two chapters discuss the representatiod of prisms, pyramids, and polyhedra, together with theis development and

intersections. The method of Monge is given the preference, but other methods are given in outline. The chapters on plane and on space curves both begin with a summary of definitions, including order, class, tangent, parametric representation, double point, osculating plane, etc. Then follow various empirical methods of constructing tangents and osculating planes. In connection with the method of central projection is found a particularly clear exposition of the relations between a space curve and its plane projection. Plücker's formulas connecting the characteristics of plane curves are presupposed, but no mention is made of the corresponding formulas for space curves. The only curve treated in detail is the cylindrical helix.

The third part, comprising about three fifths of the volume, is devoted to surfaces. As in the preceding cases, the discussion is preceded by a very condensed outline of the theory of surfaces. Three methods of representation are given; that of Monge, the central projection, and that of lines of equal height. It is shown that the first and second both contribute to the study of the apparent contour, but the third method is treated in much greater detail. The well-written chapter on surfaces of revolution employs the method of Monge almost exclusively, and is very similar to the treatment given in the better courses in descriptive geometry in our own technical schools. The discussion of helicoids is rather more analytic than graphic, and is confined to the simpler properties. The chapter on conical and cylindrical surfaces follows the usual development by the method of Monge, though a few interesting applications are made of conical projection.

Of the nine pages on developable surfaces, less than three are concerned with graphical problems; the remainder is properly devoted to a development, not merely an enumeration, of the characteristic properties of these surfaces. The discussion employs synthetic, algebraic, and analytic methods. The last chapter, on non-developable ruled surfaces, gives a briefer outline of the general theory, but proves in detail the correlation of Chasles. After showing that every algebraic ruled surface can be generated by the rectilinear transversals of three directrix curves, the graphical discussion is confined to the case in which one directrix is a straight line, in particular to helicoids and conoids.

The figures are well drawn and the type is excellent; a

reader would find the text easier to follow, however, if greater use had been made of captions, full-faced type, and indentations.

A few typographical errors occur, but most of them will not cause confusion; the only more serious ones are in the foot-note on page 203, the displayed generatrici on page 208, which should read direttrici, and the use of the word incontrano on page 209, line 20, word 6 for toccano.

The German volume is not a translation of the Italian one, although it has the same title and follows in a general way the same development. It is much more extensive, makes more extensive use of algebraic methods, has a larger number of figures, and a larger variety of displayed formulas, theorems, and problems.

The resolution of the problems of trihedra and of spherical triangles is quite similar to the Italian text; the discussion of polyhedra includes the derivation and use of Euler's formulas, and a number of exercises for the reader. But the greatest change is found in connection with the treatment of curves and surfaces. Instead of the embarrassing brevity noted above, we here find a veritable treatise, almost exclusively from the analytic standpoint. The general theory is followed by a description of several algebraic and transcendental curves, in particular the cycloids and spirals. The graphical representation by means of approximations closely follows the Italian text.

In the case of space curves formulas for curvature, torsion, and the relations between a space curve and its plane projection are derived.

The ratio of the material is about the same through the remainder of the volume. It seems odd that it was thought necessary to speak in such detail of quadric surfaces after developing so large a number of formulas for surfaces defined by any analytic function, but the outline of the treatment of the intersection of two quadrics is too brief and too difficult to be of greatest service. The chapter on cones and cylinders skillfully combines the usual graphical representation with algebraic and analytic proofs, thus removing the objection to the more frequent empirical procedure.

Ruled surfaces and developables precede surfaces of revolution; the discussion is more detailed and somewhat more

extensive, yet follows the same line as in the Italian text. The traditional treatment of surfaces of revolution follows the derivation of the general equation and of a number of characteristic properties, including illustrative examples of the ring surface.

The volume closes with a discussion of helicoids, each problem being introduced by a detailed analytic treatment.

No applications to shades and shadows or to other technical uses are made, the authors pointing out that such things would only act as a digression from the purpose of the book, which is to provide a theoretical development, suitable for teachers rather than for practitioners. A generous number of foot-notes give the origin of the important theorems and considerable other interesting information. A third volume is in preparation, which will give a systematic history of the development of the subject, and is to contain a detailed index of all three volumes.

VIRGIL SNYDER.

Vorlesungen über Differential- und Integral-Rechnung. Von EMANUEL CZUBER. Dritte, sorgfältig durchgesehene Auflage. I Band, xiv + 605 pp., II Band, x + 590 pp. 8vo. Leipzig, Teubner, 1912. 12 Mks. each.

IT is scarcely necessary to review these well known volumes at great length. Professor Czuber writes in a clear and convincing style and his treatment of the processes of the calculus and the applications is classic.

The second volume of the second edition was reviewed for the BULLETIN by the present writer, May, 1909 (volume 15, pages 392–395). No occasion has since arisen for changing the views there expressed.

The two volumes contain much more material than is ordinarily included in the elementary and advanced courses in the calculus as given in this country. The first volume contains an excellent basis for an elementary course in differential geometry and nearly one third of the second volume is devoted to differential equations and their applications.

The principal changes from the second edition may be briefly noticed. The first volume of the third edition contains a treatment of roulettes, focal lines, or caustic curves in the plane, and loxodromes on surfaces. These curves were not considered in the second edition. The first volume is also

increased in several other less important particulars; e. g., Delaunay's theorem concerning the meridians of surfaces of revolution of constant mean curvature has been added to the articles on the curvature of surfaces.

The second volume of the third edition has been increased and improved principally in the part devoted to differential equations. In particular, the section devoted to the calculus of variations has been entirely rewritten and brought into closer touch with recent work in this subject. Also a section (§ 7) has been added treating of curvilinear integrals and integrals of functions of a complex variable. The section adds but nineteen pages of new material and the treatment is limited to the outlines of the theory.

It is not too much to say, in conclusion, that the two volumes under review form an almost invaluable addition to the library of the teacher of the calculus whether from the point of view of clear and concise statement, or from that of content. It may not be out of place, in this connection, to call attention to the straightforward and rigorous treatment of the fundamental limit

$$\lim_{n=\infty} (1 + 1/n)^n = e$$

in article 30 of the first volume, in comparison with the somewhat apologetic tendency exhibited in some of our modern texts on the calculus to avoid the use of this limit. One may doubt the expediency of presenting all the details of the proof employed by Professor Czuber in a first course in the calculus, but such a doubt scarcely necessitates the use of bizarre, or non-consistent methods.

L. WAYLAND DOWLING.

Lezioni di Geometria proiettiva ed analitica. Di EDGARDO CIANI, Professore nella R. Università e nella R. Scuola Navale Superiore di Genova. Pisa, Enrico Spoerri, 1912. v+525 pp.

THE plan of replacing the traditional introductory courses in cartesian geometry and in synthetic projective geometry by one set of lectures covering the elements of both subjects is not new to Italian universities. In 1888, through the initiative of Cremona, the faculty of mathematical and physical sciences of the University of Rome sanctioned such a

course for the first year students. Texts for these courses are not so plentiful as those presenting only the elements of the classical analytical geometry. There is opportunity for diversity of opinion as to the subject matter, for obviously such a course must omit much. But a greater difficulty is that of presenting the two methods so that the student not only obtains clearly defined ideas of each but also grasps the relations between them and gains some judgment in selecting the method better suited to any particular problem. Perhaps the best known of the texts that have already appeared is that of G. Castelnuovo.*

The work under review was written especially for the engineering students, who constitute the majority of the author's hearers, at the University and the higher naval school of Genoa. One feature that distinguishes it from other texts of its kind is the absence of all mention of homogeneous coordinates. This omission is made deliberately because the author thinks that their use would be premature in the first year of study. He realizes, too, that many well drawn figures and an abundance of carefully selected exercises are necessary for an introductory course. He furnishes two hundred of the former and about three hundred of the latter.

In the first three chapters the author follows the general plan of Castelnuovo (1904–5) and A. del Re† in beginning with a chapter on the fundamental notions of projective geometry and following that by two on the analytical geometry and projectivity for forms of one dimension. The material is well chosen and is presented in a simple and attractive manner.

Plane geometry is dealt with in the next eleven chapters. The first is on cartesian coordinates, the usual equations of the straight line, distance from point to line, etc. Then the author departs from the old order of circle, parabola, ellipse and hyperbola. Instead, he attacks the general conic, beginning with a detailed treatment of its generation by two projective pencils of rays. After calling attention to the ancient Greek conception of these curves, he returns to cartesian coordinates and deduces the analytical representation of the conic. In the next chapter he studies in greater detail the principal analytical properties of the general quadratic equation in two variables and gives their geometrical inter-

* Lezioni die Geometria analitica, Roma, 1903–1909.
† Lezioni di Geometria proiettiva ed analitica, Modena, 1900.

pretation. Then follow chapters on the problem of constructing a conic when certain tangents or points are given; the theory of polarity with reference to a conic; the diameters, axes, center and foci; points common to two conics; the usual higher plane curves; projectivity for forms of two dimensions and projective geometry on a conic.

The book is concluded by eight chapters on solid geometry, —chiefly analytical.

E. B. COWLEY.

Transcendenz von e und π. Ein Beitrag zur höheren Mathematik vom elementaren Standpunkte aus. Von GERHARD HESSENBERG. Leipzig, B. G. Teubner, 1912. x+106 pp.

IT is well known that in approaching the proofs of the transcendence of e and π, either in the original form of Hermite and Lindemann, or in the simplified presentations of Hilbert, Hurwitz, and Gordan, the beginner experiences great difficulty in grasping the significance of such suddenly introduced artifices as the Hermite integral or the Hilbert polynomial.

After some introductory remarks on the "Deus ex machina" appearance of these artifices, the author presents some general reflections, abounding in pedagogical good sense, on "proofs by successive specialization" and "indirect proofs." His point of departure in presenting the proofs—which are in substance those of Hilbert, Hurwitz, and Gordan—is found in the problem of approximating the exponential function by means of the n first partial sums of its power series, each of these sums being weighted in the sense of the method of least squares. In the reviewer's opinion, this mode of presentation should prove natural and plausible to the beginner. All auxiliary propositions (on the rational and exponential functions and on algebraic numbers) are clearly and fully set forth in such a manner as not to obscure with their details the main line of thought in the transcendence proofs, the number-theoretic and analytic features of which are kept well apart.

The preceding developments lead very naturally up to a proof of Lindemann's general theorem on the non-vanishing of a linear aggregate of exponentials with unequal algebraic numbers as exponents and non-vanishing algebraic numbers as coefficients.

This little book is written in a vigorous and pleasing style, as remote from academic dryness as possible without sacrifice

of rigor, and its clever handling of a pedagogically difficult subject should recommend it to teachers of mathematics.

T. H. GRONWALL.

Vorlesungen über Variationsrechnung. Von OSKAR BOLZA. Leipzig, B. G. Teubner, 1909. ix+795+10 pp.

THIS book is a revised and considerably enlarged German edition of the same author's "Lectures on the Calculus of Variations," Chicago, University Press, 1904.*

In Chapter I, entitled "The first variation in the simplest class of problems," the author, after some introductory remarks on the scope of the calculus of variations, starts by explaining his system of notations, which is exceedingly precise and consistent, although it would seem to the reviewer that a somewhat less elaborate system would have made the book easier to read without any sacrifice of rigor. The classical results in the theory of the first variation of the integral $\int_{x_1}^{x_2} f(x, y, y')dx$ with fixed and variable end points are set forth, including Euler's differential equation and Du Bois-Reymond's lemma. The proof for the latter given on page 28 is due to Hilbert; it would perhaps have been more appropriate to give the proof of Zermelo (*Mathematische Annalen*, volume 58 (1904), page 558), which is unsurpassed in simplicity, brevity, and elegance.

Chapter II, "The second variation in the simplest class of problems," contains the Legendre and Jacobi criteria; the exposition, excellent already in the English edition, is even better in the present book and stands forth as a model of clearness and precision. Chapter III, "Sufficient conditions in the simplest class of problems," deals with the conditions for a weak minimum, the construction of a field of extremals, Weierstrass's expression for the second variation in terms of the E-function, which is here introduced by means of Hilbert's invariant integral, and various conditions for the existence of a strong minimum.

Chapter IV, "Auxiliary theorems on functions of a real variable," contains various lemmas on implicit functions and existence theorems for differential equations, preparatory to Chapter V, "Weierstrass's theory of the simplest class of problems in parametric representation," which treats anew,

* Reviewed by E. R. Hedrick in BULLETIN, vol. 12 (1906), pp. 80–90.

in parameter form, the topics dealt with in Chapters I–III besides bringing some new developments, chiefly Osgood's theorem. Chapter VI, "The case of variable end points," uses the parameter form and contains results due to Weierstrass, Kneser, and Bliss.

Chapter VII, "Kneser's theory," gives an exposition of Kneser's method of generalized geodetic coordinates, and Chapter VIII, "Discontinuous solutions," presents the results due to Weierstrass, Erdmann, and others.

Chapter IX, "The absolute extremum," proves the existence, according to Hilbert and Caratheodory, of an absolute minimum in cases where the function under the integral sign is positive for all possible values of dx/ds and dy/ds. Chapter X, "Isoperimetric problems," gives an exposition of the classical results, as well as those due to Weierstrass and Kneser, which are completed by the more recent researches of Lindeberg.

Thus far, the topics presented are all treated, although in a less complete fashion, in the English edition; the following chapters, however, are new. Of these, chapters XI, "The Euler-Lagrange multiplicator method," and XII, "Further necessary conditions, as well as sufficient ones, in the Lagrange problem," give a very complete and lucid presentation of the results hitherto obtained in the difficult problem of minimizing an integral containing n unknown functions subject to any number of accessory conditions.

Chapter XII, "Elements of the theory of the extrema of double integrals," gives a first introduction to this largely unexplored field, and the book closes with an index and an appendix containing an enumeration of the principal definitions and theorems, used throughout the book, concerning functions of a real variable.

The literature references are quite exhaustive, and the numerous exercises at the end of the chapters constitute, together with the illustrative examples given in the text, a practically complete collection of all special problems to be found in the literature since the days of Euler, thereby enhancing the value of this volume as a reference work.

To conclude: the book under review is a standard work of the highest merit, and will undoubtedly render still greater services to the investigators in this field than the English edition has already done.

T. H. GRONWALL.

Das Wissen der Gegenwart in Mathematik und Naturwissenschaft. By EMILE PICARD. Authorized German edition von F. und L. LINDEMANN. Leipzig und Berlin, B. G. Teubner, 1913. 8vo. iv+292 pp.

THIS volume has for its prime object a sweeping survey of the state of the various sciences at the beginning of the twentieth century, with the intent of making very prominent the capital importance of what the scientist calls "theories." Their significance for the advance of science and what they may rationally be expected to yield is the subject for discussion. In addition to these, the rôle that the " concept " has to play, its yielding character under the influence of new facts and of the progress of new thought, is to be kept prominently in the front as one of the main features of the discussion. Science is viewed as an asymptotic approximation towards knowledge, the convergence of which is an undemonstrable postulate and may become as slow as that of certain series. The value of science is due to the fact that it is the child of the indissoluble marriage of the beautiful and the useful. Just as Carnot in studying heat-engines founded the science of thermodynamics, and Sainte-Claire-Deville in studying platinum founded the science of chemical mechanics; or on the other hand, as Newton in writing the Principia made it possible for navigators to sail unknown seas in greater security, or Ampère and Faraday with their interest in the action of currents upon currents made the modern electric power station a possibility; so in all ages science has proceeded from two parents: the love of order and harmony in thought, and the desire for those useful inventions that would make life more comfortable and powerful.

The book is practically a translation of Picard's La Science moderne et son Etat actuel, but the Introduction has been condensed and modified, and the first chapter has had some changes. In our judgment the Introduction to the original could very profitably have been kept, inasmuch as it states in a more definite way the exact point of view from which the subject is to be examined. We will therefore extract somewhat from this original source in order to show the capital importance of the book itself for all thinkers, and particularly for those who are inclined to yield to the seductions of a certain will-o'-the-wisp, which lures them to hunt for an ultimate and completely finished philosophy of the universe.

After stating that it is not possible to go deep into the questions that arise with regard to science without analysing the manner in which we acquire knowledge, the author points out that the only source of knowledge is the flowing stream of consciousness. The phenomena separate into two classes, those which come into our consciousness through our senses and which our will is not sufficient to give birth to; and those which arise without the intermediation of the senses, as thought and memory. Thus we have the vague notions of the exterior and of the interior worlds. Through experience and the use of many organs under the action of the will, the illusions of sense are eliminated, and the residue of the exterior world we call the *real*. This involves also the appreciation of certain invariable relations between a first sensation and those we can cause to follow it. This real must also be considered as it is viewed by society, in particular by that part of society which is made up of those called *sane*. These invariant relations depend upon successive experiments of growing precision, and become the more valuable in proportion as they permit us to predict with greater certainty. Among these invariants arising from the exterior real we find certain ones that are independent of ourselves, others that depend upon us: the objective elements and the subjective elements. Among those that are objective some are constant in such a way as to give us the idea of *thing*, others are variable, like motion or position. Among the subjective elements there are some from which we abstract certain parts, forming *concepts*. These concepts are not absolutely rigid, but are subject to modification from new facts or from a sharper distinction of objective and subjective elements. The important point is that concepts are fundamental in the genesis of science, and scientific knowledge tends to development by their means, but their arbitrariness shows the presence of the human mind working upon the data of experience. In this connection logic is seen to be an admirable instrument, powerless to create, but powerful to transform concepts and bring out their unexpected consequences, thus leading us frequently to more extended concepts or to more general hypotheses. We come also to believe in laws of nature under the impulsion of our esthetic feeling and from our observation of the practical utility of such a belief. It is hopeless to attempt to demonstrate the principle of causality, but nevertheless we act as if it were true, and come to the

expected result. A system of concepts associated with laws or particular facts and transformed by convenient deductions, laid upon the framework of certain hypotheses, so as to produce a stable structure, is a scientific *theory*. The arbitrary character of the concepts is accentuated in the more arbitrary character of the hypotheses, and it often happens that the complicated network of concepts, facts, hypotheses, and reasonings of two very different theories will account equally well for a given group of phenomena. We can always make a theory agree with a set of phenomena by modifying the concepts and altering the hypotheses in the right manner. But we are under the guidance of a desire for simplicity and harmony, in all modifications.

It is thus evident that the human mind in large part creates science, and that the concepts we use and the hypotheses we connect them together with, are our own invention. And although these are conditioned in a serious way by the objective world, yet the hope of an ultimate purely objective science is a chimera, for science is measured by our own capabilities and will always be dependent upon our relations to the external world.

In consequence of the nature of science the different fields in which investigators are working must continually tend towards union with each other and the consequent revision of concepts, hypotheses, and principles. Physics and chemistry have produced physical chemistry, and the greatest suitor for the hand of the latter today is biology. One becomes a little frightened when he contemplates the breadth of learning that the investigator of the future in any one field must possess, for he may have to be acquainted with all fields of scientific learning. Even technique is becoming a complex faculty and years of practice are inevitable for the successful experimenter. Further the need of cooperation is increasing daily, so that we find mathematicians and physicists, chemists and physiologists, biologists and psychologists, all combining forces. The schools of applied science furnish not only problems but facilities to the pure scientist. In the near future it is evident that national and international cooperation and support for the advance of science not only will be seen to be desirable, but will be necessary.

It is obvious therefore that a book like this is one that no student of science should fail to read, and he ought to have

it on his shelf. We would be glad to see the day when every candidate for the doctorate would be subject to examination on such a text. We will welcome the day when one course at least is given in every university, and required of all graduate students, in this survey of all the sciences, their history, progress, and present state as to methods, principles, and correlations.

A word as to the contents of the book is all that is necessary, as no briefer account of it can be given than the text itself. The opening chapter is on the development of mathematics and its relation to the other sciences. This is followed by chapters on Mathematics and astronomy, Mechanics and energetics, The physics of the ether, The physics of matter and chemistry, Mineralogy and geology, Physiology and biological chemistry, Botany and zoology, Medicine and bacteriology. Explanatory notes by the translator close the book, and these will be welcomed by the careful student.

Anything from the pen of Professor Picard will bear the mark of that perspicuous thinker, and this difficult piece of work is no exception. Moreover the style is so charming that one finishes reading it with satisfaction that scientific exposition can still be done so perfectly.

JAMES BYRNIE SHAW.

Maxima und Minima in der elementaren Geometrie. Von RUDOLF STURM. Leipzig and Berlin, B. G. Teubner, 1910. v+138 pp.

IN this book, the author makes use of nothing but elementary geometry and trigonometry, the latter being used in proving a few of the theorems. The first two theorems in the book are arithmetical, namely:

(1) Of n positive numbers whose product is given, the sum is smallest when the numbers are equal.

(2) If the sum of n positive numbers is given, the product is greatest when the numbers are equal.

These two theorems are the basis of quite a number of proofs on areas and perimeters of polygons. When the problem under consideration can be reduced to a sum or product with the necessary restrictions, the maximum or minimum values follow readily from (1) or (2). With all the possible combinations of restrictions on the sides and angles of a triangle, the author has dealt in detail. Polygons both regular and

irregular, polygons circumscribed about a circle or inscribed in a circle, also figures bounded by straight lines and arcs of circles, all are examined in regard to their forms when their areas or perimeters have maximum or minimum values. Pairs of right triangles are also treated in a few special cases. The sum of the distances and the sum of the squares of the distances of a point from n points in a plane is minimized and some interesting conclusions are drawn. The theorem is given for the sum of the hth powers of the distances when $n = 3$. The minimum perimeter is found for a triangle inscribed in a given triangle and for a quadrilateral inscribed in a quadrilateral that is inscribed in a circle. There are several quite unexpected results in connection with this problem.

Part two treats of solid geometry. Prisms and tetrahedrons are studied in the same detail as the triangles in part one.

In most respects the book reads very smoothly. The number of very slightly different cases treated makes some parts of it almost tiresome, but the detail is no doubt justifiable. On the other hand, one finds a severe brevity of statement in some places, but these are rather few. Typographical errors are not many, but the following may be noted: near the bottom of page 15, read $U - 2c$ for $U - 2C$. In line 4 from the bottom of page 18, read "innerem" for "äusserem." Just below theorem 57, page 41, read $\rho > \rho'$. In the middle of page 47, read 60 for 69. In the first line of § 6, page 51, read 43 for 41. The value of $ABCD$ on page 73 is incorrect. In the value of $\angle\, O_1B_1O_2$, page 75, read $C_2B_1O_2$ for $B_1C_1O_2$ and CBO for BCO. In the value for A_1B_1', page 86, β_1 and β_1' are somewhat confused. On page 114, opposite figure 30, read B for B_1. On page 30, the author seems not to notice the possibility of $C + D$ being equal to 180° but this does not vitiate his conclusions.

J. V. McKelvey.

Advanced Shop Mathematics. By Earle B. Norris and Ralph T. Craigo. New York, McGraw-Hill Book Company, 1913. xi+214 pp.

This volume presents the second half of the instruction papers in shop mathematics as developed and used in the extension division of the University of Wisconsin. The intention of the authors as stated in the preface is "to present such of the principles of algebra, geometry, trigonometry and

logarithms as have been found to be of practical value in the shop, showing some of the better known applications and making the presentation as practical as possible." The statement is made that the course has proved to be "a good mathematical preparation for advanced technical study."

It has been of interest to the reviewer to note the ability with which the mathematical difficulties have been overcome by a clearness and simplicity of statement and style. Proofs are almost entirely absent. The purpose is to show how things are done. Formulas are given ex cathedra with a "this is so" accompaniment. The uses of algebra, geometry, and trigonometry are displayed to the reader, with the evident intention of arousing interest in one who has not pursued the usual elementary courses. It would seem, however, that the scope of the book would be inadequate as a basis for "advanced technical study."

As an example of text books intended for technical high schools, and as indicating the mathematical training which in the near future a number of candidates for admission to college will undoubtedly offer, the volume is of exceptional interest.

The direction to "scribe an arc" is novel and the construction given on page 84 for a flat circular arc results in an ellipse.

PERCEY F. SMITH.

Wahrscheinlichkeitsrechnung. Von E. CZUBER. Zweite Auflage in zwei Bänden. I. Band: *Wahrscheinlichkeitstheorie, Fehlerausgleichung und Kollektivmasslehre.* 1908. x+410 pp. II. Band: *Mathematische Statistik, Mathematische Grundlagen der Lebensversicherung.* 1910. x+470 pp. B. G. Teubner, Leipzig und Berlin.

THIS is the second edition of the well known work of Czuber on probability. If material for a course is wanted, possibly no better reference could be given than to these books. The first volume contains the general theory with such applications as are mainly of mathematical interest. At the end of this volume are four and seven place tables of the probability function and a four place table of its derivatives. The second volume contains the applications to questions of statistics and insurance with tables relating to those topics.

In the first volume occurs a series of problems continuously numbered and solved as applications of the theory. In this

list occur the most interesting of the classical applications. The volume begins with a discussion of probability in finite sets of events, illustrated by games of chance. This is followed by a section on geometrical probability, a subject to which the author has given considerable attention in another publication. To one who is mainly interested in the application of the theory to problems of physics the non-existence of any definite criterion for equal probability in these cases makes this the most questionable part of the subject. In games of chance the equally probable cases are fairly obvious and so one is not much surprised that the theory agrees with the average results of actual play. But in such cases as the needle problem of Buffon the agreement of experiment and theory seems mainly to prove that in the solution of the problem the choice of equally probable events has been a physically sound one. A theory justifying this choice on mechanical grounds would be interesting.

In the next section occur the theorems of Bernoulli and Poisson. In this part we would like to see applications using larger numbers, and thus approximating more nearly the conditions of atomic physics. Such illustrations as that given by Perrin, where the probable interval between two successive cases of a brick jumping of its own accord as high as the second story of a house is expressed in years by a number of billions of figures, show more clearly than these game problems how a universally observed occurrence (i. e., the failure of a brick thus to jump) may be the result of chance.

The second part of this volume contains the applications of the theory to the correction of observations. Here we have the usual discussion of the method of least squares well illustrated by tables of data.

A third part is devoted to a more recent topic, Kollektivmasslehre. This subject developed by Fechner, Lipps, and Bruns, has for its object the study of Menge of similar objects which can be arranged relatively to some numerically expressible characteristic. In this study occur a distribution function showing the relative density of the ensemble at each value of the characteristic and a sum function giving the percentage of the objects whose characteristics fall below a given value. A series is obtained expressing the sum function in terms of the probability function and its derivatives, and means are given for the calculation of the coefficients in this

series. If all the coefficients after the first are zero, the distribution follows the law of Gauss. If some of them are not zero, there is a variation from this simple law. The series usually converges very rapidly.

H. B. PHILLIPS.

Mathematical and Physical Papers. By SIR WILLIAM THOMSON, BARON KELVIN. Volume IV. *Hydrodynamics and General Dynamics.* xv+563 pp., 1910. Volume V. *Thermodynamics, Cosmical and Geological Physics, Molecular and Crystalline Theory, Electrodynamics.* xv+602 pp., 1911. Volume VI. *Voltaic Theory, Radioactivity, Electrons, Navigation and Tides, Miscellaneous.* viii+378 pp., 1911. Arranged and revised with brief annotations. By SIR JOSEPH LARMOR. Cambridge (England): at The University Press.

THAT the works of Lord Kelvin are now available in collected form is a source of gratification to physicists, mathematicians, and especially to mathematical physicists including students of mechanics. Particular thanks are due to Sir Joseph Larmor, who for these editorial duties must have sacrificed a great deal of time that could otherwise have gone to his own researches. And in the present confused state of theoretical physics we sorely need those researches.

Varied as were Lord Kelvin's contributions to physics, he may well be ranked as a student of mechanics and of the mechanical explanation of the world. The central monument in his system is the Thomson and Tait, from which he looks deep into every surrounding structure, and which itself is, to the present time, the climax of the works begun in the Principia. It was this mechanical bent which led him to search so constantly for a mechanical, as opposed to a purely electromagnetic, ether; and it was this which caused his contributions to ether theories to be less vital than those of some others who kept closer in touch with the electromagnetic point of view and who have built up the idea of the electromagnetic theory of matter and mechanical actions. In these matters Lord Kelvin during the last thirty years of his life should be classed as conservative if not reactionary.

The three volumes before us are the continuation of the series which Lord Kelvin himself had started. In 1882 Volume I appeared with seventy-three papers chiefly of dates 1841–53; two years later Volume II showed papers numbered 74 to 91,

which had appeared mainly in 1853–56; and in 1890 the third volume, dealing mainly with elasticity, heat, and electromagnetism, appeared with titles 92 to 104. There was also the separate volume, Reprint of papers on electrostatics and magnetism, which had appeared first in 1872 and again in 1884. And furthermore there are the Baltimore Lectures, delivered in 1884, but printed (with numerous additions) only in 1904. In view of this diverse procedure, Larmor found himself unable profitably to continue the old numbering of the first three volumes, and has carried a new series of numbers from 1 to 277 through the three volumes he edits. To some of these numbers correspond merely the titles of papers previously reprinted but cited here for continuity; and there are at the end of volume V some sixty pages of contributions (to engineering societies) which carry no serial number.

It would be futile here to attempt to go into detail as to the contents of these three volumes, which range over some sixty years and contain many of the important contributions of a great and versatile mathematician and physicist; to review them critically would be to review in large measure the progress of physics for the second half of the nineteenth century. We refer the reader to the subheadings, as listed above, to Larmor's interesting prefaces, and to his life of Lord Kelvin in the *Proceedings* of the Royal Society, volume 81 (1908), pages iii–lxxvi. The closer student can only be referred to the text itself of Lord Kelvin's Works.

E. B. Wilson.

Some Problems of Geodynamics, being an Essay to which the Adams' Prize in the University of Cambridge was adjudged in 1911. By A. E. H. Love. Cambridge (England), University Press, 1911. xxvii+180 pp.

It was not surprising that Love, whose treatise on Elasticity has been the standard for so long, should sometime turn his attention to the difficult applications to geodynamics. He had indeed already printed several papers upon the subject when in 1910 the selection for the Adams' Prize was "Some investigation connected with the physical constitution or motion of the earth." His winning essay is now printed.

The hypothesis upon which the author works is that of isostasy, specialized in such a way as to make it definite. Approximately the hypothesis of isostasy regards the earth

as made up of a central core in hydrostatic equilibrium surrounded by a thin "crust" in which the amount of mass in any vertical column is proportional to the base of the column; this means that under heavy high mountains there must be a relatively light region. At first blush this supposition might seem the opposite of reasonable, but it is widely adopted with a view to explaining certain anomalies in gravitational surveys.

The analytical method is that of harmonic analysis which for the most part is based upon Love's earlier discovery that the inequalities in the lithosphere may be approximately represented by an expansion containing the three surface harmonics of lowest order. This expansion does not pretend to represent the mountain ranges on the earth (indeed it makes northern Africa the highest land on the earth), but it does represent adequately the contour of the lithosphere at mean sphere level which is about two miles below sea level. It is reasonable to suppose that if the materials of the earth's crust are strong enough to support the inequalities which the assumed expansion requires, then those materials should be tenacious enough to support the great continental block as it exists. The question of the support of mountain ranges is referred to a special chapter in which the results for a harmonic of high degree (the fiftieth) are calculated.

Other topics treated are: earth tides and the effects of inertia or spheroidal figure upon them, theory of a gravitating compressible planet and the effect of compressibility on earth tides, gravitational instability, vibrations of a gravitating compressible planet, propagation of seismic waves. In the theory of a gravitating compressible planet Love had previously published a long investigation, which he now decides to replace by another founded upon a different hypothesis.

The main body of the text is a mass of intricate analysis interspersed with very readable comment. To make the points of view and the results of the work even more easy to survey, we are provided with a long and carefully written introductory abstract. If any young American student desires to assimilate the formal work of the text, he had best eschew the pure mathematician and such courses as form practically the totality of our mathematical instruction and hie him to some astronomer under whom he may learn accuracy in detailed formal analysis and safety in intricate approximation —unless he happens to have the rare good fortune to find such

a course as B. O. Peirce and W. E. Byerly used to give at Harvard on harmonic analysis and potential function.

E. B. WILSON.

Annuaire pour l'An 1914 *publié par le Bureau des Longitudes.* Paris, Gauthier-Villars. vii+502 pp., with four appendices.

THE most interesting article in the current *Annuaire* is one on the measurement of the day. In this M. G. Bigourdan has given a brief but sufficient summary of its early history and has carried it up to the present time, when the system of hour zones is fairly well established throughout the world. An interesting part of the story is his description of the efforts made in France to introduce the Greenwich meridian for the measurement of civil time and its final accomplishment in 1911. Two brief articles, one on the deformation of the images in telescopes by M. Hatt and the other on the seventeenth international geodetic congress by M. B. Baillaud, complete the "Notices."

Several revisions have been made in the astronomical portions, tending to bring the constants and descriptions up to date. But the mass of information contained in a small compass is too great for special mention here. The volume may perhaps be classified as the most complete travellers' guide to the physical universe which has hitherto been issued. There is even a slight tendency towards the inclusion of biological subjects in the tables of analyses of beers, wines, cereals, and the ashes of plants.

E. W. BROWN.

Les Actions à Distance. Par G. COMBEBIAC. Scientia No. 30. Paris, Gauthier-Villars, 1910. 89 pp.

IN this issue of *Scientia*, M. Combebiac brings together the physical foundations of some of the hypotheses which have been advanced to explain action at a distance. In all of them, a fluid is postulated: the different ways in which motions may be set up in this fluid practically constitute the developments given in the volume.

The formulas of vectors and spherical harmonics are briefly set forth—too briefly we fear for anyone not familiar with them and their uses. Two chapters are respectively devoted to the pulsating spheres of Bjerknes and the oscillating spheres of Korn with special reference to the gravitational hypotheses

which have been deduced from their properties; and finally some three chapters to the well-known analogies between hydrodynamics and electromagnetism. One chapter is somewhat of a polemic on mechanical explanations of the phenomena of physics.

There are two fundamental difficulties which are always encountered in any attempt at mechanical or electrical explanations of gravitation. The first arises from the marvellous accuracy of the law: at the distance of the moon there can be no deviation from the well-known index 2 of the inverse square greater than one divided by a hundred million, and very probably the deviation is less than one tenth of this fraction. At one time Newcomb brought forward a larger deviation to explain the outstanding difference between theory and observation in the motion of the perihelion of Mercury, but abandoned it when the theory and observation of the moon showed that its adoption would require the explanation of a much larger error in the motion of the perigee of the moon. In fact, astronomers of the present day rarely invoke any such hypothesis to explain their difficulties.

The second difficulty arises from the apparently instantaneous propagation of gravitation. It has been computed that its velocity must be at least a million times that of light. In view of these facts one is tempted to wonder whether any of the ordinary mechanical explanations are possible, unless indeed it is a case where two effects counterbalance one another, as in the principle of relativity. It must be said in this connection that M. Combebiac emphasizes in his preface the fact that analogies and not explanations constitute the chief object he has in view. "Mais l'analogie," he says, "n'est elle pas l'un des plus efficaces moyens utilisés par l'esprit humain dans son effort d'adaptation au déterminisme naturel?" For this reason, the volume will be acceptable to all those who have puzzled their minds over this fascinating problem.

E. W. Brown.

The Mechanics of the Earth's Atmosphere. A Collection of Translations (third). By Cleveland Abbe. Smithsonian Miscellaneous Collections, Vol. 51, No. 4.

The plan adopted by Professor Abbe for forwarding the interests of meteorology by the republication of the more important memoirs has several advantages peculiar to this

subject. It is not one which is ordinarily taught in educational institutions except incidentally in connection with physiography or geology, and naturally it is rare for a student to go much further than to learn the principal phenomena and their causes. If, later, he wishes to obtain a more extended grasp of the subject by reading its literature, he is confronted with a mass of discussions, opinions, guesses, and facts of every variety of value from the worst to the best. The treatises which are available are not extended enough for the sifting of all this material. Hence when a worker with long experience gathers together the papers and memoirs which have real value, translates those in a foreign tongue and arranges them in a manner sufficient to show their connection, he performs a service which cannot fail to be highly appreciated by future if not by present generations of meteorologists.

The danger in this procedure—that of tending to make future investigations follow the same grooves as those of the past—is probably not very serious. The study of the motions of our atmosphere is only beginning to emerge from the condition of a pure theory to one which has some relation to the phenomena, and it is probably better for the present to follow lines of investigation already established than to start many new methods, lest neither the old nor the new be properly developed.

The theoretical difficulties arise mainly from the fact that, in the language of the mathematician, there is apparently no first approximation; at any rate, none generally applicable has yet been found. Forces which ordinarily might or would be neglected in a first approximation have a habit of causing effects as great or greater than those which arise from the forces which have been included. The motions of liquids are troublesome enough, but they can usually be reduced to symbols unless there is turbulence: the motions of the atmosphere which we desire chiefly to know are nearly always turbulent. There is much to be done too in reducing law and order in the mass of observations which has been collected during the last few decades. A thorough investigation into the best methods of treating this material seems to be almost as much needed as is theoretical research.

From the point of view of the mathematician a detailed review of Professor Abbe's third collection is not necessary. The great majority of these papers consist of applications of

theory to observation and they are therefore somewhat outside the province of a mathematical society. It is interesting to notice, however, that while most of the papers have been published since 1880, we have one by George Hadley dated 1735. Another by Poisson (1837) is on a subject which has received attention once again, namely, the motion of projectiles taking into account the rotation of the earth.

E. W. BROWN.

NOTES.

THE April number (volume 15, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Sur la notion de différentielle d'une fonction de ligne," by M. FRÉCHET; "A type of primitive algebra," by J. H. M. WEDDERBURN; "Properties of surfaces whose asymptotic curves belong to linear complexes," by C. T. SULLIVAN; "Relatively uniform convergence of sequences of functions," by E. W. CHITTENDEN; "Note on Fermat's last theorem," by H. S. VANDIVER; "A set of axioms for line geometry," by E. R. HEDRICK and L. INGOLD; "The Cauchy problem for integro-differential equations," by G. C. EVANS.

THE March number (volume 15, number 3) of the *Annals of Mathematics* contains the following papers: "On continued fractions in non-commutative quantities," by J. H. M. WEDDERBURN; "A new type of solution of Maxwell's equations," by H. BATEMAN; "Relation between the zeros of a rational integral function and its derivate," by T. HAYASHI; "The invariants, seminvariants, and linear covariants of the binary quartic form modulo 2," by L. E. DICKSON; "Examples of normal domains of rationality belonging to elementary groups," by G. A. MILLER; "On Lebesgue's constants in the theory of Fourier series," by T. H. GRONWALL; "The linear difference equation of the first order," by K. P. WILLIAMS; "Geometric properties of the Jacobians of a certain system of functions," by A. EMCH; "On the irregular integrals of linear differential equations," by C. E. LOVE.

AT the meeting of the London mathematical society held

March 12 the following papers were read: By W. BURNSIDE, "The rational solutions of the equation $x^3 + y^3 + z^3 = 0$ in quadratic fields"; by H. HILTON and Miss R. E. COLOMB, "Orthoptic and isoptic loci of plane curves"; by G. H. HARDY, "The roots of the Riemann ζ function"; by T. J. I'A. BROMWICH, "Normal coordinates in dynamics."

THE royal Belgian academy of sciences announces the following prize problems for the year 1915:

"An important contribution to the infinitesimal geometry of curved surfaces."

"Summarize the works on systems of conics in space, and add new investigations of these systems."

Competing memoirs should be sent to the secretary before August 1, 1915. The value of each prize is 800 francs.

UNIVERSITY OF CHICAGO.—The following courses are announced for the summer quarter, 1914:—By Professor E. H. MOORE: Linear integral equations, four hours; Solid analytical geometry, four hours.—By Professor G. A. BLISS: Differential geometry, four hours.—By Professor D. R. CURTISS: Theory of functions, four hours.—By Professor L. E. DICKSON: Linear associative algebras, four hours; Theory of equations, four hours.—By Professor K. LAVES: Analytical mechanics, four hours.—By Professor F. R. MOULTON: The problem of three bodies, four hours.—By Professor H. E. SLAUGHT: Elliptic integrals, four hours.—By Professor J. W. A. YOUNG: Critical review of secondary mathematics, four hours.

DARTMOUTH COLLEGE.—The following courses are announced for the summer session, July 6 to August 15, 1914:—By Professor A. D. PITCHER: Principles of the calculus.—By Professor E. G. BILL: Solid analytical geometry; Advanced aspects of elementary geometry (primarily for teachers).—By Dr. F. M. MORGAN: Higher geometry; The teaching of elementary algebra.

THE following university courses in mathematics are announced for the present summer semester:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Elementary geometric establishment of the most important properties of conics, two hours; Space curves and surfaces,

four hours; Calculus of variations, four hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of determinants, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Analytic geometry, four hours; Theory of theta functions, four hours; Seminar, two hours.—By Professor G. HETTNER: Introduction to ordinary differential equations, two hours.—By Professor J. B. KNOBLAUCH: Integral calculus, with exercises, five hours; Applications of elliptic functions, four hours.—By Professor R. LEHMANN-FILHÉS: Analytic mechanics, with exercises, five hours.—By Dr. K. KNOPP: Differential calculus, with exercises, five hours; Infinite series, products, and continued fractions, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Theory of elliptic functions, four hours; Seminar, two hours.—By Professor F. LONDON: Descriptive geometry, with exercises, four hours; Analytic geometry, II, three hours.—By Professor I. SCHUR: Theory of numbers, two hours; Theory of Fourier series, two hours; Seminar, two hours.—By Dr. J. O. MÜLLER: Introduction to the calculus, with exercises, four hours; Introduction to geodesy, with exercises, two hours; Seminar, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor D. HILBERT: Differential equations, four hours; Selected chapters of statistical mechanics, two hours; Seminar, two hours.—By Professor E. LANDAU: Differential and integral calculus, with exercises, four hours; Theory of finite groups, four hours.—By Professor C. CARATHÉODORY: Functions of real variables, four hours; Seminar, two hours.—By Professor C. RUNGE: Numerical calculation, four hours; Exercises on graphical methods, two hours.—By Dr. E. HECKE: Theory of functions of a complex variable, four hours; Exercises in differential equations, two hours.—By Dr. L. v. SANDEN: Descriptive geometry, four hours; Technical mechanics, with exercises, three hours.—By Professor F. BERNSTEIN: Theory of probabilities, three hours; Mathematics of insurance, two hours.—By Dr. R. COURANT: Analytic geometry, with exercises, six hours.—By ——: Algebra, four hours.

UNIVERSITY OF LEIPZIG.—By Professor K. ROHN: Descriptive geometry, with exercises, four hours; Algebraic curves,

with exercises, five hours.—By Professor O. HÖLDER: Functions of a complex variable, four hours; Calculus of variations, two hours; Seminar, two hours.—By Professor G. HERGLOTZ: Plane analytic geometry, with exercises, five hours; Algebraic analysis, two hours; Seminar, two hours.—By Professor P. KOEBE: Theory of numbers, two hours; Differential equations, four hours.—By Dr. R. KÖNIG: Algebraic functions, two hours.

THE completion by Professor L. KOENIGSBERGER of the one hundredth semester of his academic career was celebrated at the University of Heidelberg on February 28.

DR. R. KÖNIG, of the University of Leipzig, has been appointed associate professor of mathematics at the University of Tübingen.

DR. W. VOGT, of the technical school at Carlsruhe, has been appointed associate professor of mathematics at the University of Heidelberg.

DR. E. A. ANSEL has been appointed docent in mathematics and astronomy at the University of Freiburg.

DR. W. BEHRENS has been appointed docent in mathematics at the University of Göttingen.

DR. G. BERNDT, of the University of Halle, has been appointed docent in mathematics at the technical school of Berlin.

DR. L. FÖPPL has been appointed docent in mathematics at the University of Würzburg.

DR. A. LACKNER has been appointed docent in geometry at the technical school of Vienna.

AT the University of Cambridge, Mr. G. H. HARDY has been appointed Cayley lecturer in mathematics, and Mr. ARTHUR BERRY a university lecturer in mathematics.

AT the University of Chicago, Professor E. J. WILCZYNSKI has been promoted to a full professorship of mathematics.

DR. T. H. GRONWALL, of Princeton University, has been appointed to an assistant professorship of mathematics at Oberlin College.

PROFESSOR H. D. THOMPSON, of Princeton University, has been granted leave of absence for the academic year 1914–15.

PROFESSOR B. H. CAMP, of Wesleyan University, has been promoted to a full professorship of mathematics.

PROFESSOR W. T. DUTTON, of Allegheny College, died March 24 at the age of 62 years.

PROFESSOR G. M. MINCHIN, formerly of the royal Indian engineering college, died March 23 at the age of 68 years.

DR. J. S. MACKAY died at Edinburgh on March 25, in his 71st year.

DR. G. W. HILL, third president of the American Mathematical Society, died at West Nyack, N. Y., on April 16, at the age of 76 years.

CHARLES S. S. PEIRCE, eminent for his researches in mathematical logic, died at Milford, Pa., on April 19, at the age of 75 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ACH (N.). Ueber die Erkenntnis a priori insbesondere in der Arithmetik. Leipzig, Quelle & Meyer, 1913.

ARMSTRONG (C. N.). Eine Untersuchung der Anwendbarkeit rekurrenter Reihen zur Aufsuchung versteckter Periodizitäten. (Diss.) München, 1913. 8vo. 96 pp.

BARTHEL (E.). Die Erde als Totalebene. Hyperbolische Raumtheorie mit einer Voruntersuchung über die Kegelschnitte. Leipzig, Hillmann, 1914. 8vo. 7+111 pp. M. 2.50

BLAHA (V.). Transzendente Zahlen, insbesondere e und π. (Progr.) Wien, 1913. 8vo. 80 pp.

CAPPELLETTI (G.). Numeri primi. Teoria e applicazioni. Cenno sui numeri perfetti e amicabili. Verona, Cinquetti, 1913. 8vo. 30 pp. L. 0.50

CAUNT (G. W.). An introduction to the infinitesimal calculus. With applications to mechanics and physics. London, Clarendon Press, 1914. 8vo. 588 pp. Cloth. 12s.

CHINI (M.). Esercizi di calcolo infinitesimale. 2a edizione, con aggiunte. Livorno, Giusti, 1914. 8vo. 10+300 pp.

CRUEWELL (E. R.). Der Satz des Fermat. 3te vermehrte Auflage. Berlin, Buschhardt, 1914. 8vo. Pp. 117–122. M. 2.40

DILLER (J. B.). Ueber die den Enneperschen Flächen konstanten negativen Krümmungsmasses entsprechenden Vossschen Flächen. (Progr.) Würzburg, 1913. 8vo. 79 pp.

EMDE (F.). See RUNGE (C.).

ENCYCLOPÉDIE des sciences mathématiques. Edition française. Tome III, volume 2: Géométrie descriptive. Géométrie élémentaire. Fascicule 1: Schoenflies-Trosse, Géométrie projective; Steinitz-Merlin, Configurations. Leipzig, Teubner, 1913. 8vo. Pp. 1–160.

ENZYKLOPÄDIE der mathematischen Wissenschaften. Band II: Analysis. 2ter Teil, 2tes und 3tes Heft: R. Fricke, Elliptische Funktionen; R. Fricke, Automorphe Funktionen mit Einschluss der elliptischen Modulfunktionen. Leipzig, Teubner, 1913. 8vo. Pp. 177–470. M. 9.20

ERB (T.). Ueber die asymptotische Darstellung der Integrale linearer Differenzen-Gleichungen durch Potenzreihen. (Diss.) München, 1913.

FERRARI (A.). Sulla somma degli angoli interni di un poligono piano convesso. Torino, Negro, 1913. 8vo. 6 pp.

GOT (T.). Questions diverses concernant certaines formes quadratiques ternaires indéfinies et les groupes fuchsiens arithmétiques qui s'y rattachent. (Thèse.) Toulouse, Privat, 1913. 4to. 105 pp.

GRAUSTEIN (W. C.). Eine reelle Abbildung analytischer komplexer Raumkurven. (Diss.) Bonn, 1913. 8vo. 107 pp.

GROSSMAN (J.). Ueber die Nullstellen der Riemannschen ζ-Funktion und der Dirichletschen L-Funktion. (Diss.) Göttingen, 1913.

HALSTED (G. B.). See POINCARÉ (H.).

HEILBRONN (J.). Die mathematischen und naturwissenschaftlichen Anschauungen des Josef Salomo Medigo dargestellt nach seinem Sefer Elim. (Diss.) Erlangen, 1913. 8vo. 92 pp.

HELLINGER (E.). See KLEIN (F.).

HERRMANN (E.). Ueber die einförmige Bewegung des ebenen kreisverwandt-veränderlichen Systems. (Diss., Dresden.) Weida i. Th., Thomas & Hubert, 1913. 8vo. 93 pp.

HILL (M. J. M.). The theory of proportion. London, Constable, 1914. 8vo. 128 pp. Cloth. 8s. 6d.

HOBSON (E. W.). "Squaring the circle." A history of the problem. Cambridge, University Press, 1913. 8vo. 66 pp. Cloth. 3s.

JAHRBUCH über die Fortschritte der Mathematik. 42ter Band (1911), 2tes Heft. Berlin, Reimer, 1913. 8vo. Pp. 497–720. M. 8.00

KAISER (A.). Weiteres zur Invariantentheorie der gewöhnlichen Differentialgleichung zweiter Ordnung. (Diss., Greifswald.) Leipzig, Teubner, 1913.

KLEIN (F.). Elementarmathematik vom höheren Standpunkte aus. 2ter Teil: Geometrie. Vorlesungen, gehalten 1908. Ausgearbeitet von E. Hellinger. 2te Auflage. Leipzig, Teubner, 1914. 8vo. 8+547 pp. M. 7.50

——. Lectures on the icosahedron and the solution of equations of the fifth degree. Translated by G. G. Morrice. 2d revised edition. London, K. Paul, 1914. 8vo. 10s. 6d.

MARCHAND BEY (E. E.). Géométrie plane. Quelques nouveautés intéressantes. Nouvelles méthodes de détermination de la vraie valeur de π, etc. Vannes, Lafolye. 8vo. 36 pp.

MIKAMI (Y.). See SMITH (D. E.).

MORRICE (G. G.). See KLEIN (F.).

PANHÖLZL (V.). Wie man im 17. Jahrhundert das Problem von der Quadratur des Zirkels zu lösen suchte. (Progr.) Budweis, 1913. 8vo. 15 pp.

PASCH (M.). Veränderliche und Funktion. Leipzig, Teubner, 1914. 8vo. 6+186 pp. Cloth. M. 7.00

POINCARÉ (H.). The foundations of science: science and hypothesis, the value of science, science and method. Authorized translation by G. B. Halsted, with a special preface by Poincaré, and an introduction by J. Royce. (Science and education.) New York, Science Press, 1913. 8vo. 11+553 pp. Cloth. $3.00

RAUBER (A.). Ueber die Lösung der Differentialgleichungen, welche die Bewegung des dreiachsigen Kreisels um einen festen Punkt beschreiben. (Diss.) München, 1913.

ROHRAUER (G.). Die Epi- und Hypozykloiden als einhüllende Kurven. (Progr.) Olmütz, 1913. 8vo. 12 pp.

RUNGE (C.) und EMDE (F.). Rechnungsformular zur Zerlegung einer empirisch gegebenen periodischen Funktion in Sinuswellen. Mit Erläuterung von C. Runge. Braunschweig, Vieweg, 1913.

SMITH (D. E.) and MIKAMI (Y.). A history of Japanese mathematics. Chicago, Open Court Co., 1914. 8vo. $3.00

STRUISTE (L.). Die linearen diophantischen Gleichungen. (Progr.) Innsbruck, 1913. 8vo. 28 pp.

STUDY (E.). Die realistische Weltansicht und die Lehre vom Raume. Geometrie, Anschauung und Erfahrung. (Die Wissenschaft. 54ter Band.) Braunschweig, Vieweg, 1914. 8vo. 10+145 pp. M. 5.20

THALER (H.). Beleuchtungskonstruktionen an Kreiszylinderflächen. (Progr.) Stockerau, 1913. 8vo. 16 pp.

VALLÉE POUSSIN (C. J. DE LA). Cours d'analyse infinitésimale. Tome I. 3e édition, considérablement remaniée. Paris, Gauthier-Villars, 1914. 8vo. 9+452 pp. Fr. 14.00

VANORSTRAND (C. E.). Tables of the exponential functions e^x, e^{-x}, for $x = 0.0$ to $x = 32.0$, to either 20 decimals or 20 significant figures. (*Journal of the Washington Academy of Sciences*, Volume III.) Washington, 1913.

II. ELEMENTARY MATHEMATICS.

BOURLET (C.). Algèbre. Paris, Hachette, 1913. 16mo. 226 pp. Cartonné. Fr. 2.00

——. Eléments de géométrie. Avec la collaboration de P. Baudoin. 2e édition. Paris, Hachette, 1914. 16mo. 354 pp. Cartonné. Fr. 3.00

BRUNO (G. M.). Geométria y nociones de agrimensura, levantamento de planos y nivelación. 3e edición, revisada y aumentada. Paris, Procuraduria général. 16mo. 392 pp.

——. Nociones elementales de geométria aplicadas al dibujo lineal. 16e edición. Paris, Procuraduria général. 16mo. 112 pp.

BURNISTON (A.). A common-sense algebra. An elementary course, with answers. London, Heinemann, 1914. 8vo. 176 pp. Cloth. 2s.

CRANTZ (P.). Ebene Trigonometrie. (Aus Natur und Geisteswelt. Nr. 431.) Leipzig, Teubner, 1914. 8vo. 4+98 pp. Cloth. M. 1.25

CREPAS (E.). Corso di matematica. Parte VI, per la terza classe normale. Roma, A. Manzoni, 1913. 8vo. 107 pp.

DEAKIN (R.). A new school geometry. London, Mills & Boon, 1914. 8vo. 316 pp. Cloth. 2s. 6d

DENNIS (T.). An algebra for preparatory schools. Cambridge, University Press, 1914. 8vo. 164 pp. Cloth. 2s.

DOBBS (W. J.). A school course in geometry. London, Longmans, 1914. 8vo. 450 pp. Cloth. 3s. 6d.

ELÉMENTS d'arithmétique avec quelques principes de théorie. 7e édition. Paris, Vitte, 1913. 16mo. 496 pp.

FERVAL (A.). Eléments de trigonométrie. 7e édition revisée. Paris, Hachette, 1914. 16mo. 303 pp. Cartonné. Fr. 2.50

GRÉVY (A.). Eléments d'algèbre. Classes de 3e A à 1re A et B. 8e édition. Paris, Vuibert, 1913. 16mo. 280 pp.

——. Géométrie. Classes de 5e B à 3e B. 9e édition. Paris, Vuibert, 1914. 16mo. 323 pp. Fr. 2.25

——. Géométrie dan l'espace. Classes de 1er C et D. 5e édition. Paris, Vuibert, 1914. 8vo. 164 pp.

——. Géométrie plane. Classes de 2e C et D. 6e édition. Paris, Vuibert, 1914. 8vo. 280 pp. Fr. 3.00

JACQUET (E.) et LACLEF (A.). Cours de géométrie théorique et pratique. 3e édition, revue et augmentée. Paris, Nathan, 1913. 16mo. 512 pp.

MAIR (D. B.). Exercises in mathematics, with answers. London, Macmillan, 1913. 8vo. 482 pp. Cloth. 4s. 6d.

MOREY (C. W.). Advanced arithmetic. New York, Scribner, 1913. 12mo. 10+470 pp. Cloth. $0.65

MYERS (G. W.) and others. The point of view in "Second-year mathematics for secondary schools." Chicago, University of Chicago, 1912. 12mo. 12 pp.

PAHL (F.). Geschichte des naturwissenschaftlichen und mathematischen Unterrichts. Leipzig, Quelle und Meyer, 1913. 8vo. M. 10.60

PENSA (A.). Elementi di geometria. 2a edizione, riordinata e accresciuta. Prefazione di C. Burali-Forti. Torino, Gallizio, 1914. 16mo. 198 pp. L. 2.00

REDGROVE (H. S.). Experimental mensuration; an elementary text-book of inductive geometry. New York, Van Nostrand, 1913. 12mo. 17+328 pp. $1.25

RIBONI (G.). Elementi di geometria. 8a edizione, con modificazioni ed aggiunte. Bologna, Zanichelli, 1913. 16mo. 374 pp. L. 2.80

ROSENBURG (F.). Tutorial algebra. Key. 4th edition. London, Clive, 1914. 8vo. 310 pp. Cloth. 4s. 6d.

RÜEFLI (J.). Lehrbuch der Stereometrie. 4te vermehrte und verbesserte Auflage. Bern, Francke, 1914. 8vo. 6+150 pp. Cloth. M. 2.20

SMITH (D. E.). Problems in the teaching of secondary mathematics. (Address delivered before the New England Association of Teachers of Mathematics.) Boston, Ginn, 1913.

SOCCI (A.) et TOLOMEI (G.). Elementi di matematica; libro di testo. Volume I. 8a edizione. Firenze, Le Monnièr, 1913. 8vo. 212 pp. L. 1.50

TIMERDING (H. E.). See TROST (W.).

TOLOMEI (G.). See SOCCI (A.).

TRAUTWINE (J. C.). Table of four-place logarithms from Trautwine's civil engineers' pocket-book. Philadelphia, Trautwine, 1914.

TROST (W.). Die mathematischen Fächer an den niederen gewerblichen Lehranstalten in Deutschland. Mit einem Vorwort von H. E. Timerding und einem Anhang von A. Haese. (Abhandlungen über den mathematischen Unterricht. 4ter Band, 5tes Heft.) Leipzig, Teubner, 1914. 8vo. 6+150 pp. M. 4.00

WROBEL (E.). Uebungsbuch zur Arithmetik und Algebra. 1ter Teil: Pensum der Tertia und Untersekunda. 22te Auflage. Rostock, Koch, 1913. 8vo. 12+356 pp. M. 3.75

ZIMMERMANN (H.). Rechentafel nebst Sammlung häufig gebrauchter Zahlenwerte: 7te Auflage. Ausgabe A. Berlin, Ernst, 1913. 8vo. 34+204 pp. Cloth. M. 5.00

III. APPLIED MATHEMATICS.

ATTWOOD (E. L.) and COOPER (I. C. G.). A text-book of laying off; or the geometry of shipbuilding. New York, Longmans, 1914. 8vo. 13+118 pp. Cloth. $1.75

BARTON (S. M.). Elements of plane surveying, including leveling. Revised edition. Boston, Heath, 1913. 8vo. 8+255 pp. Cloth. $1.50

BAUDINE (L.). L'examen de géomètre-arpenteur. 2e partie. Ixelles-Bruxelles, Baudine, 1913. 123 pp. Fr. 4.00

BAULINO (C.). L'entropia e i diagrammi entropici; con molte applicazioni numeriche. Livorno, Giusti, 1913. 8vo. 6+90 pp.

BEAUFILS (H.). See HARANG (F.).

BLAINE (R. G.). Hydraulic machinery, with an introduction to hydraulics. 3d edition, revised. London, Spon, 1913. 8vo. 482 pp. 10s. 6d.

BLUDAU (A.). Die Kartenprojektionen in elementarer Behandlung. Düsseldorf, 1913. M. 1.80

BOREL (E.). Introduction géométrique a quelques théories physiques. Paris, Gauthier-Villars, 1914. 8vo. 7+141 pp. Fr. 5.00

BOUASSE (H.). Cours de physique. Tome III: Electricité et magnétisme. 2e édition en 3 volumes. 1re partie: Etude du champ magnétique. Paris, Delagrave, 1914. 8vo. 20+448 pp. Broché. Fr. 16.00

BRANLY (E.). Traité de physique. 5e édition. Paris, Gigord, 1914. 8vo. 29+884 pp.

——. Problèmes de physique. Enoncés et solutions. 3e édition. Paris, Gigord, 1914. 18mo. 184 pp.

BREHMER (K.). Kollineare und andere graphische Rechentafeln für geodätische Rechnungen. Stuttgart, 1913. 8vo. 5 pp. M. 2.00

BREW (W.). Three-phase transmission. 2d edition, revised and enlarged. London, Lockwood, 1913. 8vo. 190 pp. Cloth. 7s. 6d.

BUNET (P.). See STEINMETZ (C. P.).

CALDWELL (F. C.). Electrical engineering problems. New York, McGraw-Hill, 1914. 12mo. 105 pp. Cloth. $1.00

CARY (E. R.). Solution of railroad problems by the slide rule. London, Constable, 1913. 18mo. 4s. 6d.

CASTELLS Y VIDAL (P.). Las representaciones mecanicas de los fenomenos electricos. Barcelona, 1913. 4to. 29 pp. Fr. 3.75

CHAMPION (E.). Principes généraux d'électricité théorique et pratique. 2e partie. Lille, Danel, 1914. 4to. 334 pp.

COHN (E.). Physikalisches über Raum und Zeit. 2te Auflage. Leipzig, Teubner, 1913. 8vo. 24 pp. M. 0.80

COOPER (I. C. G.). See ATTWOOD (E. L.).

DINGLER (H.). Die Grundlagen der angewandten Geometrie. Leipzig, Akademische Verlagsgesellschaft. 8vo. 160 pp.

ENCYCLOPÉDIE des sciences mathématiques. Edition française. Tome IV, volume 6: Balistique-Hydraulique. Fascicule 1: C. Crantz–E. Vallier, Balistique extérieure; C. Crantz–C. Benoit, Balistique intérieure; F. Gossot–R. Liouville, Développements concernant quelques recherches de balistique exécutée en France; T. Forchheimer–A. Boulanger, Hydraulique. Leipzig, Teubner, 1913. 8vo. Pp. 1–192. M. 7.20

FOSTER (H. A.). Electrical engineer's pocket-book. 7th edition, revised. New York, VanNostrand, 1913. 16mo. 36+1599 pp. Cloth. $5.00

GEHLEN (K.). Querstabilität und Seitensteuerung von Flugmaschinen. (Diss.) Aachen, 1913.

GRAHAM (J.). An elementary treatise on the calculus, for engineering students. 4th edition. (Finsbury technical manual.) London, Spon, 1914. 8vo. 12+355 pp. 5s.

GUARRERA (P.). Corso di nautica astronomica teorico e pratico. Volume I: Elementi di cosmografia e d'astronomia sferica. Palermo, Mirto, 1914. 8vo. 15+179 pp. L. 4.50

GYLLENBERG (W.) und WICKSELL (S.). Die analytische Lösung des Bahnbestimmungsproblems in ihren numerischen Anwendungen. (*Arkiv för Mathematik.*) Stockholm, 1913. 8vo. 68 pp. M. 1.80

HANSEN (L. F.). Trigonometry and navigation. London, J. Brown, 1914. 8vo. 292 pp. Cloth. 10s. 6d.

HARANG (F.) et BEAUFILS (H.). Notions élémentaires de géométrie descriptive appliquée au dessin. 2e édition, augmentée. Paris, Dunod et Pinat, 1914. 8vo. 8+188 pp. Fr. 2.50

HARTMANN (W.). Das Maschinengetriebe: 1ter Band: Die geometrische Bewegungslehre. Stuttgart, Deutsche Verlagsanstalt, 1913. M. 20.00

HEEREN (J. 's). Manuel du bornage, à l'usage des géomètres-arpenteurs. Tirlemont, 's Heeren, 1913. 166 pp. Fr. 3.50

HERTZ (H.). Gesammelte Werke. 2ter Band: Untersuchungen über die Ausbreitung der elektrischen Kraft. 3te Auflage. Leipzig, Barth, 1914. 8vo. 7+296 pp. M. 7.50

JAROSCH (J.). Methodik des Unterrichts in der darstellenden Geometrie und im geometrischen Zeichnen. Wien, Pichler, 1913. M. 3.00

JEANS (J. H.). Gravitational instability and the nebular hypothesis. (Royal society.) London, Dulau, 1914. 4to. Sewed. 1s. 6d.

KŘIŽWANEK (K.). Analytische Darstellung der Ungleichheiten in der Bewegung des Mondes. Teschen, Prochaska, 1913. 8vo. 31 pp. M. 1.00

LABROUSTE (H.). See WOOD (R. W.).

LAMB (H.). Dynamics. Cambridge, University Press, 1914. 8vo. 356 pp. Cloth. 10s. 6d.

LYLE (T. R.). On the self-inductance of circular coils of rectangular section. (Royal society.) London, Dulau, 1914. 4to. Sewed. 1s.

LYON (W. V.). Problems in alternating current machinery. New York, McGraw-Hill, 1914. 12mo. 135 pp. Cloth. $1.50

MAURER (E. R.). Technical mechanics. 3d edition, rewritten. New York, Wiley, 1914. 8vo. 350 pp. Cloth. $2.50

MOORE (H. F.). The strength of I-beams in flexure. London, Chapman & Hall, 1913. 8vo. Sewed. 1s.

NOTES et formules de l'ingénieur: Mathématiques, mécanique rationnelle, résistance des matériaux, etc. 17e édition, entièrement refondue. Paris, Geisler, 1913. 8vo. 2756 pp. Cartonné toile. Fr. 15.00

OLSSON (P.). Partikulära Integraler till Differentialekvationerna for fasta Kroppars Rörelse i Vätskor. (*Arkiv för Matematik.*) Stockholm, 1913. 8vo. 34 pp. M. 1.20

PICKELS (C. W.) and WILEY (C. C.). Text-book on railroad surveying. New York, Wiley, 1913. 16mo. 9+263 pp. Cloth. $2.50

PIÉRARD (E.). Cours d'électricité et de magnétisme. Tome I: Théories générales. 3e édition. Paris, Dunod et Pinat, 1913. 8vo. 402 pp. Fr. 11.00

PRANDTL (L.). Abriss der Lehre von der Flüssigkeits- und Gasbewegung. Jena, Fischer, 1913. M. 1.60

PRÁŠIL (F.). Technische Hydrodynamik. Berlin, Springer, 1913. M. 9.00

RENNER (J.). Stereographische Azimutal-Distanzkarten. (Progr.) Graz, 1913. 8vo. 9 pp.

RINNE (F.). Allgemeine Kristallographie und Mineralogie. (Die Kultur der Gegenwart.) Leipzig, Teubner, 1913. 8vo. Cloth. M. 20.00

RIPPER (W.). Heat engines. (Being a new edition of "Steam.") New edition. New York, Longmans, 1914. 8vo. 10+352 pp. $1.10

SENSEL (G. v.) Elektrizität und Optik, behandelt vom Standpunkte der Elektronentheorie. Wien, 1913. M. 0.90

SOTHERN (J. W. M. and R. M.). Elementary mathematics for marine engineers. London, Munro, 1913. 8vo. 174 pp. Cloth. 2s. 6d.

SPATH (F.). Theorie des elektrischen Gleichgewichts in elementarer Behandlung. (Progr.) Salzburg, 1913. 8vo. 51 pp.

STEINMAN (D. B.). Suspension bridges and cantilevers, their economic proportions and limiting spans. 2d edition, revised. (VanNostrand's Science Series.) New York, VanNostrand, 1913. 24mo. 9+7+185 pp. $0.50

STEINMETZ (C. P.). Théorie et calcul des phénomènes électriques de transition et des oscillations. Traduit par P. Bunet. Paris, Dunod et Pinat, 1912. 8vo. 10+578 pp. Fr. 22.00

STELLERSCHE Vorrichtung zu gleichzeitiger, verbundener Darstellung helio- und geozentrischer Planetenbewegungen. Nürnberg, Steller, 1913.

TRANSTROM (H. L.). Electricity at high pressures and frequencies. Chicago, Branch Publishing Co., 1913. 12mo. 247 pp. Cloth. $2.00

USHERWOOD (T. S.) and TRIMBIE (C. J. A.). Practical mathematics for technical students. Part 1. London, Macmillan, 1914. 8vo. 376 pp. Cloth. 3s. 6d.

VÂLCOVICI (V.). Ueber diskontinuirliche Flüssigkeitsbewegungen mit zwei freien Strahlen. (Diss.) Göttingen, 1913.

VAUCHER (A.). Théorie mathématique de l'échelle musicale. Paris, Gauthier-Villars, 1913. 8vo. 68 pp.

VOS (M.). Ueber eine neue Form der Stosserregung elektrischer Schwingungen. (Diss.) Marburg, 1913. 8vo. 50 pp.

WELLS (M. B.). Steel bridge designing. Chicago, M. C. Clark, 1913. 8vo. 8+260 pp. Cloth. $2.50

WERTHEIMER (E.). Elektromagnetische Theorie der Dämpfe mit spezieller Berücksichtigung des Wasserdampfes. Zürich, 1913. M. 2.00

WICKSELL (S.). See GYLLENBERG (W.).

WIESNER (S.). Grundlinien der Kinematik. Wien, 1913. M. 0.85

WILEY (C. C.). See PICKELS (C. W.).

WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. 2ter Band: Festigkeitslehre. 2te verbesserte Auflage. Berlin, 1913. M. 6.00

WOOD (R. W.). Optique physique. Traduit de l'anglais, d'après la 2me édition, par H. Labrouste. Tome 2: Etude des radiations. Paris, Gauthier-Villars, 1914. 8vo. 484 pp. Fr. 18.00

WRIGHT (H. B.). Transmission line formulas. London, Constable, 1913. 8vo. 8s. 6d.